UNIT - I

Mathematical logic

Introduction :-

Logic is the signs dealing with the method of reasoning plays a very imp role in every area of knowledge, particularly mathematics, logic expressed in such a way, note as Symbolic logic (or) mathematical logic

Statement (or) proposition :-

* A proposition is declarative sentence which in a given content can be said to be either true (or) false but not both
* Propositions are usually represented by small letters p, q, r, s ....
* The truth (or) false of proposition is called the "truth value". If the proposition is true we will indicate by "1" and if it is false we indicate by "0"

Logical connectivities and truth tables :-

The combine of simple statements to form a compound statement by using certain connecting words known as "connectivities".

when the primary statements are combined by means of connectivities "not, and, or, if, iff" they are

* Negation ($\sim$ or $\neg$)
* Conjunction ($\wedge$)
* Disjunction ($\vee$)
* Conditional / Implication ($\rightarrow, \Rightarrow$)
* Bi- Conditional / Bi- Implication ($\leftrightarrow, \Leftrightarrow$)

Negation :-

* A proposition obtained by inserting the word "not" (or) "no" at an appropriate place in a given proposition is

called the "negation of given proposition".

P	~P
T	F
F	T

Eg:- p: Tirupati is in ts (false)
~p: Tirupati isnot in ts (True)

Conjunction :-

* A compound proposition obtain by combining two given propositions by inserting the word "and" in b/w them & it is called the "conjunction of given propositio

P	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction :-

* The compound proposition obtained by combining two given propositions by inserting the word " or" in b/w. them it is called "disjunction of given proposition"

P	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Conditional / Implication :-

* Let p, q is a propositions then the conditional $p \Rightarrow q$ is the proposition i.e., false when p is true and q is false otherwise true. ($\rightarrow$ or $\Rightarrow$)

P	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional / Bi Implication :-

Let p, q be the propositions then the biconditional is the proposition i.e., true for p & q have same truth and false values otherwise false. ($\leftrightarrow$ (or) $\Leftrightarrow$)

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \Rightarrow q$	$p \Leftrightarrow q$	$p \vee q$
T	F	F	F	T	T	T	T
T	F	F	T	F	F	F	T
F	T	T	F	F	T	F	T
F	F	T	T	F	T	T	F

i) $(p \wedge q) \wedge q$ (ii) $(p \vee q) \wedge p$ (iii) $\sim p \Leftrightarrow \sim q$ (iv) $p \sim p$ (v) $q \sim q$
vi) $(\sim p) \Leftrightarrow \sim q$

i)

p	q	$p \wedge q$	$(p \wedge q) \wedge q$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

ii)

p	q	$p \vee q$	$(p \vee q) \wedge p$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

iii)

p	q	$\sim p$	$\sim q$	$\sim p \Leftrightarrow \sim q$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

iv)

p	$\sim p$
T	F
F	T

v)

q	$q \sim p$
T	F
F	T

vi) $\neg p \Rightarrow \neg q$

p	q	$\neg p$	$\neg q$	$\neg p \Rightarrow \neg q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

i) $p \oplus q$ ii) $\neg p \oplus q$

i)

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

ii)

p	q	$\neg p$	$\neg p \oplus q$
T	T	F	T
T	F	F	F
F	T	T	F
F	F	T	T

a) $p \vee \neg p$ b) $p \wedge \neg p$ c) $(p \vee \neg p) \Rightarrow q$ d) $(p \vee q) \Rightarrow (p \wedge q)$

e) $p \Rightarrow q$ f) $\neg p \Leftrightarrow q$ g) $(p \Rightarrow q) \Leftrightarrow (\neg p \Rightarrow \neg q)$

h) $(p \Rightarrow q) \Rightarrow (q \Rightarrow p)$

a)

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

b)

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

e)

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

c)

p	q	$\neg p$	$p \vee \neg p$	$(p \vee \neg p) \Rightarrow q$
T	T	F	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	T	F

d)

p	q	$p \lor q$	$p \land q$	$(p \lor q) \Rightarrow (p \land q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

e)

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

f)

p	q	$\sim p$	$\sim p \Leftrightarrow q$
T	T	F	F
T	F	F	T
F	T	T	T
F	F	T	F

g)

p	q	$\sim p$	$\sim q$	$p \Rightarrow q$	$\sim p \Rightarrow \sim q$	$(p \Rightarrow q) \Leftrightarrow \sim p \Rightarrow \sim q$
T	T	F	F	T	T	T
T	F	F	T	F	T	F
F	T	T	F	T	F	F
F	F	T	T	T	T	T

h)

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \Rightarrow (q \Rightarrow p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

a) $(p \Leftrightarrow q) \lor (q \Leftrightarrow p)$

b) $(p \rightarrow q) \oplus (q \rightarrow p)$

c) $(p \land q) \lor (\sim p \land q) \lor (p \land \sim q) \lor (\sim p \land \sim q)$

d) $(p \land q) \oplus (\neg p \land q) \lor (\neg p \land \neg q)$.

c)

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim p \wedge q$	$(p \wedge q) \vee (\sim p \wedge q)$	$p \wedge \sim q$
T	T	F	F	T	F	T	F
T	F	F	T	F	F	F	T
F	T	T	F	F	T	T	F
F	F	T	T	F	F	F	F

$\sim p \wedge \sim q$	$(p \wedge \sim q) \vee (\sim p \wedge \sim q)$	$[(p \wedge q) \vee (\sim p \wedge q)] \vee [(p \wedge \sim q) \vee (\sim p \wedge \sim q)]$
F	F	T
F	T	T
F	F	T
T	T	T

d)

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim p \wedge q$	$(p \wedge q) \oplus (\sim p \wedge q)$
T	T	F	F	T	F	T
T	F	F	T	F	F	F
F	T	T	F	F	T	T
F	F	T	T	F	F	F

$\neg p \wedge \neg q$	$[(p \wedge q) \oplus (\sim p \wedge q)] \vee (\neg p \wedge \neg q)$
F	T
F	F
F	T
T	T

a)

p	q	$p \Leftrightarrow q$	$q \Leftrightarrow p$	$(p \Leftrightarrow q) \vee (q \Leftrightarrow p)$
T	T	T	T	T
T	F	F	F	F
F	T	F	F	F
F	F	T	T	T

b)

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \oplus (q \rightarrow p)$
T	T	T	T	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	F

Propositional equivalence :-

Toutology :-

A Compound propositional always true is called "toutology."

Contradiction :-

A compound propositional always false is called "Contradiction"

Contingency :-

A compound propositional neither a toutology nor Contradiction is known as "Contingency."

1. S.T for any two propositions p, q

a) $(p \oplus q) \vee (p \leftrightarrow q)$ is a toutology.

b) $(p \oplus q) \wedge (p \leftrightarrow q)$ is a Contradiction.

c) $(p \oplus q) \wedge (p \rightarrow q)$ is a Contingency.

a)

p	q	$p \oplus q$	$p \leftrightarrow q$	$(p \oplus q) \vee (p \leftrightarrow q)$
T	T	F	T	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

$\therefore (p \oplus q) \vee (p \leftrightarrow q)$ is a tautology.

b)

p	q	$p \oplus q$	$p \leftrightarrow q$	$(p \oplus q) \wedge (p \leftrightarrow q)$
T	T	F	T	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	F

$\therefore (p \oplus q) \wedge (p \leftrightarrow q)$ is ~~not~~ a contradiction.

c)

p	q	$p \oplus q$	$p \rightarrow q$	$(p \oplus q) \wedge (p \rightarrow q)$
T	T	F	T	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

$\therefore (p \oplus q) \wedge (p \rightarrow q)$ is a contingency.

Logical equivalence :-

* $p \wedge T \equiv p$
  $p \vee F \equiv p$ — Identity laws

* $p \vee T \equiv T$
  $p \wedge F \equiv F$ — Domination laws.

* $p \vee p \equiv p$
  $p \wedge p \equiv p$ — Idepotent laws.

* $\sim(\sim p) \equiv p$ — double negation laws.

* $p\vee q \equiv q\vee p$
  $p\wedge q \equiv q\wedge p$ — Commutative laws.

* $(p\vee q)\vee q \equiv p\vee(q\vee r)$
  $(p\wedge q)\wedge r \equiv p\wedge(q\wedge r)$ — Associative laws.

* $p\vee(q\wedge r) \equiv (p\vee q)\wedge(p\vee r)$
  $p\wedge(q\vee r) \equiv (p\wedge q)\vee(p\wedge r)$ — distributive laws.

* $\sim(p\wedge q) \equiv \sim p\vee\sim q$
  $\sim(p\vee q) \equiv \sim p\wedge\sim q$ — Demorgan's laws

* $p\vee(p\wedge q) \equiv p$
  $p\wedge(p\vee q) \equiv p$ — Absorption laws.

* $p\vee\sim p \equiv T$
  $p\wedge\sim p \equiv F$ — Negation laws.

Logical equivalence involving implications :-

* $p\to q \equiv \sim p\wedge q$
* $p\to q \equiv \sim q\to\sim p$
* $p\vee q \equiv \sim p\to q$
* $\sim(p\to q) \equiv p\wedge\sim q$
* $(p\to q)\wedge(p\to r) \equiv p\to(q\wedge r)$
* $(p\to r)\wedge(q\to r) \equiv (p\vee q)\to r$
* $(p\to q)\vee(p\to r) \equiv p\to(q\vee r)$
* $(p\to q)\vee(q\to r) \equiv (p\wedge q)\to r$

Logical equivalence involving Bi-conditional :-

* $p\leftrightarrow q \equiv (p\to q)\wedge(q\to p)$
* $p\leftrightarrow q \equiv (p\wedge q)\vee(\sim p\wedge\sim q)$
* $p\leftrightarrow q \equiv \sim p\leftrightarrow\sim q$
* $\sim(p\leftrightarrow q) \equiv p\leftrightarrow\sim q$

* S.T each of the following implications is a tautology without using truth table.

a) $(p \wedge q) \rightarrow q$ b) $p \rightarrow (p \vee q)$ c) $\neg p \rightarrow (p \rightarrow q)$

d) $(p \wedge q) \rightarrow (p \rightarrow q)$ e) $\neg(p \rightarrow q) \rightarrow p$.

Rough

p	q	$p \wedge q$	$(p \wedge q) \rightarrow q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

a)

Let the hypothesis p & q written.

→ In this case ~~p and~~ $(p \wedge q) \rightarrow q$ are true.

→ If the hypothesis is false then it is irrespective Conclusion

So, our Condition $(p \wedge q) \rightarrow q$ is always true.

∴ ~~our~~ $(p \wedge q) \rightarrow q$ is a tautology.

Rough

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

b)

Let the hypothesis p & q written.

→ In this case ~~p~~ $p \rightarrow (p \vee q)$ are true.

→ If the hypothesis is false then it is irrespective conclusion

So, our condition $p \rightarrow (p \vee q)$ is always true.

∴ $p \rightarrow (p \vee q)$ is a tautology.

Rough

p	q	$\neg p$	$p \rightarrow q$	$\neg p \rightarrow (p \rightarrow q)$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

c) Let the hypothesis p & q written

→ In this case $\neg p \rightarrow (p \rightarrow q)$ are true.

→ If the hypothesis is false then it is irrespective conclusion

So, our condition $\neg p \rightarrow (p \rightarrow q)$ is always true.

∴ $\neg p \rightarrow (p \rightarrow q)$ is a tautology.

Rough

p	q	$p \wedge q$	$p \rightarrow q$	$(p \wedge q) \rightarrow (p \rightarrow q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

d)

Let the hypothesis p & q written.

→ In this case $(p \wedge q) \rightarrow (p \rightarrow q)$ are true.

→ If the hypothesis is false then it is irrespective Conclusion

So, our Condition $(p \wedge q) \rightarrow (p \rightarrow q)$ is always true.

∴ $(p \wedge q) \rightarrow (p \rightarrow q)$ is a tautology.

e) Let the hypothesis p & q written

→ In this case $\neg(p \rightarrow q) \rightarrow p$ are true.

→ If the hypothesis is false then it is irrespective

p	q	$p\to q$	$\neg(p\to q)$	$\neg(p\to q)\to$
T	T	T	F	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

Rough

Conclusion :

So, all condition $\neg(p\to q)\to p$ is always true

$\therefore$ $\neg(p\to q)\to p$ is a tautology.

2. S.T $\sim(p\lor(\sim p\land q))$ & $\sim p\land\sim q$ are logical equivalence.

Now,

$$\sim[p\lor(\sim p\lor q)] \equiv \sim p\land\sim(\sim p\land q) \quad \text{< Demorgan's law.}$$
$$\equiv \sim p\land(\sim(\sim p)\lor\sim q) \quad \text{< demorgan's law.}$$
$$\equiv \sim p\land(p\lor\sim q) \quad \text{< double negation law.}$$
$$\equiv (\sim p\land p)\lor(\sim p\land\sim q) \quad \text{< distributive law.}$$
$$\equiv F\lor(\sim p\land\sim q) \quad \text{< negation law.}$$
$$\equiv (\sim p\land\sim q)\lor f \quad \text{< Commutative law.}$$
$$\equiv \sim p\land\sim q \quad \text{< Identity law.}$$

3. S.T $p\to(q\to r)$ & $(p\land q)\to r$ are logical equivalence

Now,
$$p\to(q\to r) \equiv \sim p\lor(q\to r) \quad < p\to q\equiv\sim p\lor q$$
$$\equiv \sim p\land(\sim q\lor r) \quad \text{< Associative law.}$$
$$\equiv (\sim p\lor\sim q)\lor r \quad \text{< demorgan's law.}$$
$$\equiv \sim(p\land q)\lor r$$
$$\equiv (p\land q)\to r \quad < p\to q\equiv\sim p\lor q.$$

4. S.T $\sim p\to(q\to r)$ & $q\to(p\lor r)$ are logical equivalence.

Now,
$$\sim p\to(q\to r) \equiv \sim(\sim p)\lor(q\to r) \quad < p\to q\equiv\sim p\lor q.$$
$$\equiv p\lor(q\to r) \quad \text{< double negation law.}$$
$$\equiv p\lor(\sim q\lor r) \quad \text{< Associative law.}$$
$$\equiv (p\lor\sim q)\lor r \quad \text{< Commutative law.}$$
$$\equiv (\sim q\lor p)\lor r \quad \text{< Associative law.}$$
$$\equiv \sim q\lor(p\lor r) \quad < p\to q\equiv\sim p\lor q.$$
$$\equiv q\to(p\lor r).$$

* S.T $\sim p\to(q\to r)$ and $q\to(p\lor r)$ are logical equivalence.

Now,
$$\sim p\to(q\to r) \equiv \sim(\sim p)\lor(q\to r) \quad \because p\to q\equiv\sim p\lor q$$

$\equiv p \vee (q \rightarrow r)$ ∵ double negation law.

$\equiv p \vee (\neg q \vee r)$ ∵ associative law

$\equiv (p \vee \neg q) \vee r$ ∵ commutative law

$\equiv (\neg q \vee p) \vee r$ ∵ Associative law

$\equiv \neg q \vee (p \vee r)$ ∵ $p \rightarrow q \equiv \neg p \vee q$

$\equiv q \rightarrow (p \vee r)$.

* Find ~~composition~~ compound proposition of :-

a) $(p \wedge q) \vee (\neg p \wedge q) \wedge (p \wedge \neg q) \vee (\neg p \wedge \neg q)$.

b) $(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)$.

b)

p	q	r	$p \vee q$	$p \rightarrow r$	$(p \vee q) \wedge (p \rightarrow r)$	$q \rightarrow r$	$[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)]$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	F	F
F	F	T	F	T	F	T	F
F	F	F	F	T	F	T	F

a)

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg p \wedge q$	$(p \wedge q) \vee (\neg p \wedge q)$	$p \wedge \neg q$	$\neg p \wedge \neg q$
T	T	F	F	T	F	T	F	F
T	F	F	T	F	F	F	T	F
F	T	T	F	F	T	T	F	F
F	F	T	T	F	F	F	F	T

$(p\wedge\neg q)\vee(\neg p\wedge\neg q)$	$[(p\wedge q)\vee(\neg p\wedge q)]\wedge[(p\wedge\neg q)\vee(\neg p\wedge\neg q)]$
F	F
T	F
F	F
T	F

Connecting Nand and Nor:-

* the compound proposition $\sim(p\wedge q)$ is denoted by $p\uparrow q$. The symbol '$\uparrow$' is called the Nand connectivity.
* The compound proposition $\sim(p\vee q)$ is denoted by $p\downarrow q$. The symbol '$\downarrow$' is called the Nor connectivity.

Nand

$(\wedge)$ T, T $\to$ T

$(\uparrow)$ T, T $\to$ F

Nor

$(\vee)$ F, F $\to$ F

$(\downarrow)$ F, F $\to$ T

* Find truth table for $p\uparrow q$ and $p\downarrow q$.

p	q	$p\uparrow q$	$p\downarrow q$.
T	T	F	F
T	F	T	F
F	T	T	F
F	F	T	T

* S.T 1) $\sim(p\downarrow q)\equiv(\sim p\uparrow\sim q)$.
  2) $\sim(p\uparrow q)\equiv(\sim p\downarrow\sim q)$.

1)

p	q	$\sim p$	$\sim q$	$p\downarrow q$	$\sim(p\downarrow q)$	$\sim p\uparrow\sim q$.
T	T	F	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	T	F	F

2)

p	q	$\sim p$	$\sim q$	$p \uparrow q$	$\sim(p \uparrow q)$	$\sim p \downarrow \sim q$
T	T	F	F	F	T	T
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	T	F	F

Predicate and quantifiers :-

* Declare two sentence lyk $x>5$, $x=y+3$, $x+y=z$ are not propositions unless the symbols x,y,z are specified sentences of this kind is called open statement.

→ The statement is $x>3$, Here x = subject
>3 = predicate

* It has two part are subject and predicate it is denoted by $P(x)$ wheer 'p' is predicate and 'x' is variable.

→ let $p(x)$ is said to be values of the propositional function p at x.

Ex:-1

Let $p(x)$ denoted by the statement $x>3$. what are the truth values of $P(4)$ & $P(2)$.

G.T :-

$p(x) = x>3$

Now,

$p(4) = 4>3$. It is true.

Also $p(2) = 2>3$. It is false.

Ex:-2

Let $Q(x,y)$ denotes the statement $x=y+3$, wt are the truth values of $Q(1,2)$ and $Q(3,0)$.

G.T :-

$Q(x,y)$ is equals $x=y+3$ $\underline{\quad}$ ① and $Q(1,2)$

Here $x=1$, $y=2$

from ① $x=y+3$

$1=2+3$

$1=5$

It is false

Also $Q(3,0)$

Here $x=3$, $y=0$

from ① $x=y+3$

$3=0+3$

$3=3$

It is true

Ex:-3

What are the truth values of the propositions are $R(1,2,3)$ and $R(0,0,1)$ if $R(x,y,z)$ denotes $x+y=z$.

G.T :-

$R(x,y,z)$ denotes $x+y=z$

Now $R(1,2,3)$

Here $x=1$, $y=2$, $z=3$

Since $x+y=z$

$1+2=3$

$3=3$

$\therefore$ It is true.

Also $R(0,0,1)$

Here $x=0$, $y=0$, $z=1$

$\therefore$ $x+y=z$

$0+0=1$

$0=1$

$\therefore$ It is false.

NOTE :-

* In general statement involving 'n' variables are $x_1, x_2, x_3, \ldots, x_n$ can be denoted by $p(x_1, x_2, \ldots x_n)$.

Compound statement in predicate logic :-

* Rama is a teacher and his teaching is good.

$T(R) \wedge G(T)$

* Rama is a teacher ~~and~~ his teaching is good.

$T(R) \vee G(T)$

* Rani is a teacher then her teaching is good.

$T(R) \Rightarrow G(T)$.

Quantifiers :-

* Quantifiers are two types

→ Universal quantifier [$\forall$]

→ Existential Quantifier [$\exists$]

Universal Quantifiers :-

The universal quantifiers of $p(x)$ is the proposition of $p(x)$ is true or false $\forall$ values of 'x'. in the universe of discuss and it is denoted by $\forall x\, p(x)$.

Here $\forall$ is called the universal Quantifier.

Ex:-1

Let $p(x)$ denotes $x+1>x$ wt is the truth values of the quantification [for all ($\forall$)] $\forall x\, p(x)$ where the universe is the set of real numbers.

Since $p(x) = x+1>x$ is true $\forall$ real no.s

The quantification is $\forall x\, p(x)$ always true.

Ex:-2

Let $G(x)$ denotes $x<2$, wt is the truth value of $\forall x\, p(x)$ where the universe is in the set of real no.s

Since $G(x) = x<2$

At $x=3$

$G(3) = 3 < 2$

It is false.

$\therefore \forall x G(x)$ is false.

NOTE :-

* when all the elements in the universe of discuss can listed by say $x_1, x_2 \dots x_n$ then $\forall x P(x) = P(x_1) \wedge P(x_2) \wedge \dots \wedge P(x_n)$.

Universe of discuss:-

Many mathematical statements assert that a property is true or false $\forall$ the values variables in a particular domain is called "universe of discuss"

Ex :- 3

what is the truth value of $\forall x p(x)$ where $p(x)$ denotes $x^2 < 10$ and the universe of discuss is the +ve integer no exceeding 4

G.T :- $P = \{1, 2, 3, 4\}$.

$p(x) = x^2 < 10$

$p(1) = 1^2 < 10$

$= 1 < 10$ It is true.

$p(2) = 2^2 < 10$

$= 4 < 10$. It is true.

$p(3) = 3^2 < 10$.

$= 9 < 10$. It is true.

$p(4) = 4^2 < 10$

$= 16 < 10$ It is false.

$\forall x p(x) = p(1) \wedge p(2) \wedge p(3) \wedge p(4)$

$\forall x p(x)$ is false

Ex :- 4

Wt does the statement $\forall x p(x)$ means if $T(x)$ is 'x' has parents and the universe of discuss consists of all people.

Every people has 2 parents, this statement is always true.

$\therefore \forall x p(x)$ is true.

Uses of universal Statements :-

For, each, and every one, anything, everything, nothing and for any.

Existential Quantifiers [$\exists$] :-

The existential Quantifier of $p(x)$ is [the proposition of $p(x)$] true or false $\exists$ values of 'x' and its denoted by $\exists x p(x)$.

Here '$\exists$' is called existential quantifier.

Uses of existential Quantifiers Statements :-

There exists, Some, there is atleast one and there exists some.

Ex:-1

Let $p(x)$ denotes the statement $x>3$, wt is the truth value of the quantification $\exists x p(x)$ where the universe of discuss consists of all real no.s

G.T :- $p(x) = x > 3$

$p(4) = 4 > 3$ It is true.

Ex:-2

Let $Q(x)$ denotes the statement $x = x+1$, wt is the truth values of $\exists x Q(x)$ where the universe of discuss is consist of all real no.s

G.T :-

$Q(x)$ denotes $x = x+1$

At $x=1$

$\because x = x+1$

$1 = 1+1$

$1 = 2$ It is false

At $x=5$

$\because x = x+1$

$5 = 5+1$

$5 = 6$ It is false

$\therefore \exists x Q(x)$ is false.

NOTE:-

* If all elements in the universe of discuss can be listed by $x_1, x_2 \ldots x_n$ is equals to $\exists x p(x) = p(x_1) \vee p(x_2) \vee \ldots \vee p(x_n)$

Ex:-3

wt is the truth value of $\exists x p(x)$ where $p(x)$ is the statement

$x^2 > 10$ and universe of discuss consists of the +ve integers not excluding 4^2.

G.T :-

$p(x) = x^2 > 10$ $\quad P = \{1^2, 2^2, 3^2, 4^2\}$

$p(1) = 1^2 > 10$
$= 1 > 10$ It is false.

$p(2) = 2^2 > 10$
$= 4 > 10$ It is false

$p(3) = 3^2 > 10$
$= 9 > 10$ It is false

$p(4) = 4^2 > 10$
$= 16 > 10$ It is true.

$\therefore \exists x\, p(x) = p(1^2) \vee p(2^2) \vee p(3^2) \vee p(4^2)$. It is ~~false~~ true.

Nested Quantifiers :-

A quantifiers that occur within the scope of other quantifiers is called a "nested quantifiers."

Ex:-1

Translate following statements into English lang. where $C(x)$ is Comedians, $F(x)$ is funny and universe of discuss is all people

(i) $\forall x[C(x) \to F(x)]$ (ii) $\forall x[C(x) \wedge F(x)]$ (iii) $\exists x[C(x) \to F(x)]$

(iv) $\sim\forall x[C(x) \vee F(x)]$

(i) All comedian people are funny
(ii) All people are Comedians and funny
(iii) There comedian people who is funny.
(iv) Not all people are comedians or funny

Ex:-2

Translate following statements into english where $R(x)$ is rabbit, $H(x)$ is a Hare and universe of discuss is all animals

a) $\forall x[R(x) \to H(x)]$ b) $\forall x[R(x) \wedge H(x)]$ c) $\exists x[R(x) \to H(x)]$

d) $\sim\forall x[R(x) \vee H(x)]$.

a) All rabbit animals are hare.
b) All animals are rabbits & hare.
c) There rabbit animals who is hare.
d) Not all animals are rabbit or hare.

Ex:-3

Translate the Statement $\forall x, \forall y, (x>0) \land (y<0) \rightarrow (xy<0)$ into english. where the universe of discuss is real no.s and for both x, y are real no.s

The given statement says that x, y are real no.s if x is +ve and y is -ve then xy is -ve. The product of +ve and -ve real no. is a -ve real no.

Ex:-4

Translate the statement $\forall x \forall y \forall z, f(x,y) \land f(x,z) \land f(y \neq z) \rightarrow f(y,z)$. in-to English where $f(x,y)$ means frnds and universe of discuss for x, y, z consists of all students in our clg.

The given statement says for x, y, z are all students if $f(x,y)$ are frnds. and $f(x,z)$ are also frnds and $f(y \neq z)$ aren't frnds then $f(y,z)$ are frnds.

Negating nested Quantifiers :-

* wt is the negation of the expression $\exists x p(x)$?

G.T :- $\exists x p(x)$ $[\because \sim\exists = \forall]$

Now, $\sim[\exists x p(x)] = \forall x \sim p(x)$.

* wt is the negation of $\forall x p(x)$?.

G.T :- $\forall x p(x)$ $[\because \sim\forall = \exists]$

Now,

$\sim[\forall x p(x)] = \exists x \sim p(x)$.

* Express the negation of the statement $\forall x \exists y (xy=1)$

G.T :- $\forall x \exists y (xy=1)$

Now,

$\sim[\forall x \exists y (xy=1)]$

$= \exists x \forall y \sim(xy=1)$.

$= \exists x \forall y (xy \neq 1)$

The order of Quantifiers :-

* Let $p(x,y)$ be the Statement $x+y=y+x$. what is the truth value of the quantification $\forall x \forall y\, p(x,y)$.

Let $p(x,y)$, $x+y=y+x$

$\therefore$ The property is true, $\forall$ real no. x,y ---- Quantification. $\forall x \forall y$ $p(x,y)$ is true.

* Let $q(x,y)$ denote $x+y=0$. wt are the truth value of Quantification $\exists y \forall x\, q(x,y)$ & $\forall x \exists y\, Q(x,y)$.

Let $q(x,y)$, $x+y=0$. The quantification $\exists y \forall x\, q(x,y)$ denotes the proposition, there is a real no. y such that for every real no. x, but there is no real no. y such that $x+y=0$ $\forall$ real no. x.

IIly $\forall x \exists y\, q(x,y)$ denotes for every real no. x there is a real no. y such that $x+y=0$ since for every x there is $y=-x$. Such that $x+y=0$

$x-x=0$ is true.

$\exists y \forall x\, q(x,y)$. It is false.

* Let $q(x,y,z)$ be the statement $x+y=z$. wt are the truth values of the statement $\forall x \forall y \forall z \exists q(x,y,z)$ & $\exists z$ $\forall x \forall y\, q(x,y,z)$.

Let $q(x,y,z)$, $x+y=z$

Then $\forall x \forall y \forall z \exists \neg q\, (x,y,z)$ $\forall$ real no. x & $\forall$ real no. y $\forall$ real no. z, there is no real no. z such that $x+y=z$.

IIly $\exists z \forall x \forall y (x,y,z)$ there is a real no. z such that $\forall$ real no. x $\forall$ real no. y. $x+y=z$ is a false.

Rule of inference	Tautology	Name
$\begin{array}{c} p \\ \hline \therefore p\vee q \end{array}$	$p \to (p\vee q)$	Addition
$\begin{array}{c} p\wedge q \\ \hline \therefore p \end{array}$	$(p\wedge q) \to p$	simplification
$\begin{array}{c} p \\ q \\ \hline \therefore p\wedge q \end{array}$	$(p\wedge q) \to (p\wedge q)$	conjunction..
$\begin{array}{c} p \\ p\to q \\ \hline \therefore q \end{array}$	$[p\wedge(p\to q)]\to q$	modus ponens
$\begin{array}{c} \sim q \\ p\to q \\ \hline \therefore p \end{array}$	$[\sim q\wedge(p\to q)]\to \sim p$	modus tollens.
$\begin{array}{c} p\to q \\ q\to r \\ \hline p\to r \end{array}$	$[(p\to q)\wedge(q\to r)]\to (p\to r)$	hypothetical Hypothetical syllogism
$\begin{array}{c} p\vee q \\ \sim p \\ \hline \therefore q \end{array}$	$[(p\vee q)\wedge \sim p]\to q$	disjunctive syllogism
$\begin{array}{c} p\vee q \\ \sim p\vee r \\ \hline q\vee r \end{array}$	$[(p\vee q)\wedge(p\vee r)]\to (q\vee r)$	Resolution.

Rules of inference :-

* we will now introduce rules of inferences for Propositional logic. These rules provides the justification of steps used to show that a conclusion follows logically from a set of hypothesis. The following table has import rules of inferences

1) State which rule of inferences is the basis of the following argument it is below freezing now ∴ it is either below freezing or raining now.

Let p is the proposition

p: It is below freezing now.

q: It is Raining now.

$$\frac{p}{\therefore p \vee q}$$

p	q	$p \vee q$	$p \to p \vee q$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

which is the addition rule it is tautology.

2) state which rule of inferences is the basis of the following argument It is below freezing and Raining now ∴ therefore it is below freezing now.

p : It is below freezing

q : Raining now.

$$\frac{p \wedge q}{\therefore p}$$ It is simplification. It is tautology.

3) State which rule of inferences is the basis of the following argument. If it rains today then we will not have a barbeque today, if we do not have a barbeque today then we will have barbeque tomorrow therefore if it rains today then we will have a barbeque tomorrow.

p: If it rains today.

q: we will not have a barbeque today.

r : we will have a barbeque tomorrow.

$p \to q$

$q \to r$ It is hypotical Syllogism. It is tautology.

p	q	r	$p\to q$	$q\to r$	$p\to r$	$(p\to q)\wedge(q\to r)$	$(p\to q)\wedge(q\to r)\to(p\to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

NOTE:-

* Consider a set proposition $P_1, P_2, P_3 \dots P_n$ then a Compound proposition $P_1 \wedge P_2 \wedge P_3 \wedge \dots \wedge P_n \to q$ Here '$P_1, P_2, P_3 \dots P_n$' are called 'premises' of the argument 'q' is called 'Conclusion' of the argument.

→ the premises $P_1, P_2 \dots P_n$ of an argument is said to be in Consistent if their Conjunction $P_1 \wedge P_2 \wedge P_3 \dots \wedge P_n$ is false. In every posible situation.

→ the premises $P_1, P_2 \dots P_n$ of an argument is said to be Consistent if their Conjunction $P_1 \wedge P_2 \dots \wedge P_n$ is True. In atleast one possible situation.

Ex:-

i) $p\wedge(\sim p\wedge q)$ Consistent or inconsistent.

the conjunction of all premises $p\wedge(\sim p\wedge q)$ is false in all cases

$\therefore$ $p\wedge(\sim p\wedge q)$ is inconsistent.

p	q	$\sim p$	$\sim p\wedge q$	$p\wedge(\sim p\wedge q)$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	F
F	F	T	F	F

ii) $(p\vee q)\wedge\sim p$.

In premises of $(p\vee q)\wedge\sim p$ in all cases result is false one case it is true.

$\therefore (p \vee q) \wedge \sim p$ it is Consistent

p	q	$\sim p$	$p \vee q$	$(p \vee q) \wedge \sim p$
T	T	F	T	F
T	F	F	T	F
F	T	T	T	T
F	F	T	F	F

3) S.T the hypothesis it is not sunny this afternoon AND it is coolder then yesterday. we will go swimming only if it is sunny If we do not go swimming then we will take a canoe trip and if we take a canoe trip then we will be home by sunset lead to the conclusion we will be home by sunset.

p : It is sunny this afternoon
q : It is cooler than yesterday.
r : we will go swimming.
s : we will take a canoe trip.
t : we will be home by sunset.

Then the hypothesis become $(\sim p \wedge q), (r \to p), (\sim r \to s) \wedge (s \to t)$

Steps	Reasons
$\sim p \wedge q$	hypothesis
$\sim p$	Simplification using (1)
$r \to p$	hypothesis.
$[\sim p \wedge (r \to p)] \to \sim r]$	
$\sim r$	modus tollens 2 & 3
$\sim r \to s$	$[\sim r \wedge (\sim r \to s)] \to s$
s	$s \to t$.

4) S.T the hypothesis if you send me an email msg then I'll finish writing the program if you don't send me an email message then I will go to sleep early and if I go to sleep early then I will wakeup feeling refresh lead to the conclusion. If I don't finish writing the program then I will wakeup feeling refreshed.

p : you send me an email msg.
q : I will finish writing the program
r : I will go to sleep early.
s : I will wakeup feeling refresh.

Then the hypothesis $(p \to q)$ $(\sim p \to r) \wedge (r \to s)$ $(\sim q \to s)$.
this argument shows that the hypothesis 'p' to be
Conclusion

Steps	Reasons
$p \to q$	hypothesis
$\sim q \to \sim p$	Contrapositive (1)
$(\sim p \to r)$	hypothesis
$\sim q \to r$	hypothetical (2) & (3) $[\because (\sim q \to \sim p) \wedge (\sim p \to r) \to \sim q \to r]$
$r \to s$	hypothesis
$\sim q \to s$	hypothetical 4 & 5 $[\because (\sim q \to r) \wedge (r \to s) \to \sim q \to s]$

Direct proof :-

the implication $p \to q$ can be proof by showing that p is true then q must be true, this shows that p is true and q is false never occur. A proof of this kind is called direct proof.

1. show that direct proof of statement is product of 2 odd integers is also odd integer.

let $x = 2n+1$
$y = 2m+1$.
$x \cdot y = (2n+1)(2m+1)$.
$= 4mn + 2n + 2m + 1$.
$= 2(2)mn + 2n + 2m + 1$.
$= 2(2mn + m + n) + 1$.
$= 2a + 1$.

2. show that square of an even no. is also an even no.

$x = 2n$
S.O.B.S
$x^2 = (2n)^2$

$= 4n^2$

$= 2(2n^2)$

$= 2a.$

3. Show that sum of 2 odd no. is an ~~4 & 5~~ even no.

Let $x = 2n+1$

$y = 2m+1.$

$x+y = (2n+1) + (2m+1).$

$= 2n + 2m + 2$

$= 2(n+m+1)$

$= 2a.$

Indirect proofs :-

$p \to q$ in this we assume that q is false then by logical argument we arrive at situation when $\sim q \to q$. contradiction this can happen only when $\sim q$ is false which implies that q must be true.

Ex:- 1

p : $3n+2$ is an odd no.

q : n is odd no. show that p is even and odd.

Sol:-

Assume that q is odd no.

$\sim q$ is even

The statement n is even $(2k)$

Given $3n+2 = 3(2k)+2$

$= 6k+2$

$= 2(3k+1)$

$= 2a$

Since n is odd $(2k+1)$

Since $3n+2 = 3(2k+1)+2$

$= 6k+3+2$

$= 6k+5$

2) S.T $x \in R$ if $x^3+4x=0$ then $x=0$.

$$x^3 + 4x = 0$$

$$x(x^2 + 4) = 0.$$

$$x = 0 \text{ (or) } x^2 + 4 = 0.$$

$$\therefore x = 0.$$

3) S.T $\sqrt{2}$ is an irrational no. $\neg q$: $\sqrt{2}$ is an irrational no. q: $\sqrt{2}$ is an irrational no.

$$\sqrt{2} = \frac{x}{y}$$

S.O.B.S

$$2 = \frac{x^2}{y^2}$$

$$x^2 = 2y^2$$

$\therefore x^2$ is an even & x is also even i.e., $x = 2n$

$$(2n)^2 = 2y^2$$

$$4n^2 = 2y^2$$

$$y^2 = 2n^2$$

$\therefore y^2$ is also even no. then y is also even no. since x & y both are even numbers.

Vacuous proofs:-

Suppose that the hypothesis 'p' is false then the implication $p \to q$ is true. Because this statement is at the form $F \to t$ or $f \to f$ and it is true. This type of proof is called vacuous proof.

Ex:-

1) S.T the proposition $p(0)$ is true then $p(0) = 0 > 1$ then $0 > 0$ & $p(n)$ = If $n > 1$ then $n^2 > n$.

$p(n)$: If $n > 1$ then $n^2 > n$

$p(0)$: If $0 > 1$ then $0 > 0$.

$$f \to T$$

$\therefore$ since the hypothesis $0 > 1$ is false the implication $p(0)$ is automatically true.

Trivial proof :-

Suppose the conclusion is true then $p \to q$ is true because the statement in the form $t \to t$ (or) $f \to T$ which is true this type of proof is called trivial proof.

Ex :-

1) Let $p(n)$ if a & b are +ve integer with $a \geq b$ and then $a^n \geq b^n$. S.T the proposition $p(0)$ is true.

Given :-

$p(n)$: If a & b, $a \geq b$ then $a^n \geq b^n$.

Then $p(0) = a \geq b$ then $a^0 \geq b^0$.

$a \geq b$.

$a^0 = 1$, $b^0 = 1$

$\therefore$ $a^0 = 1$ & $b^0 = 1$ the conclusion of $p(0)$ is true.

Proof By cases :-

To prove an implication the form $(P_1 \vee P_2 \vee P_3 \cdots \vee P_n) \to q$ the tautology $\exists [(P_1 \vee P_2 \vee \cdots \vee P_n)]$ of inference this shows that $(P_1 \vee P_2 \vee \cdots P_n) \to q$ can be used as the rule of inference. This type of argument is called proof by cases.

Ex :-

1) Using proof by cases S.T $|xy| = |x||y|$. [Let x, y are real no.s]

In our proof of this theorem we remove absolute values using the fact that & $|a| = -a$ when $a \leq 0$ because both $|x|$ & $|y|$ occur. In our formula we will need four cases.

i) x & y both non-negative.
ii) x not negative y is negative.
iii) x is negative y is not negative.
iv) x & y both negative.

case (i):-

we see that $P_1 \to q$ because $xy \geq 0$ when $x \geq 0, y \geq 0$ such that $|xy| = xy = |x||y|$.

i) $P_1 \to q$ because $xy \geq 0$ when $x \geq 0, y \geq 0$ such that $|xy| = xy = |x||y|$.

ii) $P_2 \to q$ $x \geq 0$ & $y \leq 0$ such that $|xy| = x|-y| = |x||-y| = |x||y|$.

iii) $P_3 \to q$ $x \leq 0$ & $y \geq 0$ such that $|xy| = |-x|y = |x||y| = |x||y|$.

iv) $P_4 \to q$ $x \leq 0$ & $y \leq 0$ such that $|xy| = |-x||-y| = |x||y|$.

Proof of equivalence :-

To prove a theorem that is a biconditional statement is of the form $p \Leftrightarrow q$ where p and q are propositions is a tautology $p \Leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$.

Ex:-

i) Let $p: n$ is odd $q: n^2$ is odd show that the proof of equivalence is true.

Given theorem is of the form $p \to q$ to prove this theorem we need to show $(p \to q)(q \to p)$ is true.

If already proved that $p \to q$ is true. $q \to p$ is true.

$\therefore$ the theorem $p \Leftrightarrow q$ is true.

Existing Proof :-

The propositions of the form $\exists x p(x)$ is true. If a universe of discuss such that only one element is zero a universe of discuss such that $p(0)$ is true this type of proof is called existing proof.

Ex:- P.T m and n and m+n are all perfect squares.

Let $m = 9, n = 16$

which are square values.

Now, $m+n = 9+16 = 25 = (5)^2$

It is a perfect square.

UNIT - II

Induction and Recursion

Mathematical Induction :-

* Let $p(n)$ is a propositional function on +ve integers.

* In general mathematical induction can be used to prove statements that $p(n)$ is true $\forall$ +ve integers n- $n \in z^*$

* A Proof by mathematical Induction has two parts

→ Basic step

→ Inductive step.

* we show that $p(1)$ is true and $\forall$ +ve integers $p(k)$ is true then $p(k+1)$ is true.

principle of mathematical Induction :-

* To P.T $p(n)$ is true. for all +ve integers 'n' where $p(n)$ is a propositional function we complete 2 steps

* **Basic Step :-**

we verify that $p(1)$ is true.

* **Inductive Step :-**

we show that conditional statement.

$p(k) \to p(k+1)$ is true $\forall$ +ve integers.

Note :-

* Expressed as a rule of inference this mathematical Induction technique can be started as

$[p(1) \wedge \forall k[p(k) \to p(k+1)]] \to \forall n\, p(n)$.

1. S.T $1+2+3+\dots+n = \frac{n(n+1)}{2}$, if 'n' is a +ve integer.

let
$$p(n) = 1+2+3+\dots+n = \frac{n(n+1)}{2}$$

Basic part :-

If $n=1$

$$p(1) = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

$\therefore$ p(1) is true.

Inductive part :-

If $n=k$

$$p(k) = 1+2+3+\dots+k = \frac{k(k+1)}{2}$$

Now,

$$p(k+1) = 1+2+3+\dots+k+(k+1) = \frac{k(k+1)}{2}+(k+1).$$

$$= \frac{k(k+1)+2(k+1)}{2} \Rightarrow \frac{(k+1)(k+2)}{2}$$

$$\Rightarrow \frac{(k+1)[(k+1)+1]}{2}$$

$\therefore$ $p(k) \to p(k+1)$ is true.

2 S.T $1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$, if 'n' is +ve integer.

Let
$$p(n) = 1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Basic part :-

If $n=1$

$$p(1) = \frac{1(1+1)(2(1)+1)}{6} = \frac{2(3)}{6} = \frac{6}{6} = 1.$$

$\therefore$ p(1) is true.

Inductive part :-

If $n=k$

$$p(k) = 1^2+2^2+3^2+\dots+k^2 = \frac{k(k+1)(2k+1)}{6}$$

Now,

$$p(k+1) = 1^2+2^2+3^2+\dots+k^2+(k+1)^2 = \frac{k(k+1)(2k+1)}{6}+(k+1)^2.$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$= \frac{(k+1)[2k^2+k+6k+6]}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$12k^2$

$4k \quad 3k$

$$= \frac{(k+1)[2k^2+4k+3k+6]}{6} \Rightarrow \frac{(k+1)[2k(k+2)+3(k+3)]}{6}$$

$$\Rightarrow \frac{(k+1)[(k+2)(2k+3)]}{6} \Rightarrow \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

$\therefore p(k) \to (p(k+1))$ is true.

3. S.T $1^3+2^3+3^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$ if 'n' is +ve integer.

Let

$$p(n) = 1^3+2^3+3^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$$

Basic part :-

If $n=1$

$$P(1) = \frac{(1)^2(1+1)^2}{4} = \frac{4}{4} = 1$$

$\therefore$ P(1) is true.

Inductive part :-

If $n=k$

$$P(k) = 1^3+2^3+3^3+\cdots+k^3 = \frac{k^2(k+1)^2}{4}$$

Now

$$p(k+1) = 1^3+2^3+3^3+\cdots+k^3+(k+1)^3 = \frac{k^2(k+1)^2}{4}+(k+1)^3.$$

$$\Rightarrow \frac{k^2(k+1)^2+4(k+1)^3}{4}$$

$$\Rightarrow \frac{(k+1)^2[k^2+4(k+1)]}{4} \Rightarrow \frac{(k+1)^2[k^2+4k+4]}{4}$$

$$\Rightarrow \frac{(k+1)^2[k^2+2k+2k+4]}{4} \Rightarrow \frac{(k+1)^2[k(k+2)+2(k+2)]}{4}$$

$$\Rightarrow \frac{(k+1)^2[(k+2)(k+2)]}{4} \Rightarrow \frac{(k+1)^2(k+2)^2}{4}$$

$$\Rightarrow \frac{(k+1)^2[(k+1)+1]^2}{4}$$

$\therefore p(k) \to p(k+1)$ is true

4. Use mathematical Induction to P.T $2^n < n!$ for every +ve integer n with $(n \geq 4)$

Let

$p(n) = 2^n < n!$

Basic part :-

If $n = 4$

$p(4) = 2^4 < 4!$

$= 16 < 24$

$\therefore p(4)$ is true.

Inductive part :-

If $n = k$

$p(k) = 2^k < k!$

Now $p(k+1)$

$= 2^k < k! \Rightarrow \cancel{2^k \cdot 2 < (k+1)k!}$

$\cancel{\Rightarrow 2^k \cdot 2 < (k+1)((k+1)-1)!}$

$\cancel{\Rightarrow 2^{k+1} < (k+1)!}$

$\Rightarrow 2^k \cdot 2 < 2 \cdot k!$ $\quad\quad\quad < 2 < (k+1)$

$\Rightarrow 2^k \cdot 2 < (k+1)k!$ $\quad\quad\quad < n! = n(n-1)!$

$\Rightarrow 2^k \cdot 2 < (k+1)[(k+1)-1]!$

$\Rightarrow 2^{k+1} < (k+1)!$

$\therefore p(k) < p(k+1)$ is true.

5) Use M.I to prove that $n^3 < n$ is divisible by 3 if n is a +ve Integer.

Let

$p(n) = n^3 < n = n^3 - n < 0$

Basic part :-

If $n = 2$

$P(2) = 2^3 - 2 = 8 - 2 = 6$. It is divisible by 3.

$\therefore p(2)$ is true.

Inductive part :-

If $n = k$

$p(k) = k^3 - k$ is divisible by 3.

$p(k) = \frac{k^3 - k}{3}$

Now $p(k+1) = \frac{k(k^2+3k+3)}{3}$

$= \frac{k^3+3k^2+3k}{3} \Rightarrow \frac{k^3 + \cancel{1^3} + 3k^2(1) + 3k(1)^2 - k - \cancel{1}}{3}$

$\Rightarrow \frac{(k+1)^3 - (k+1)}{3}$

$\therefore p(k) \to p(k+1)$ is true.

6. Use M.I to P.T $n < 2^n$ for all Integers $n \geq 1$.

Let

$p(n) = n < 2^n$

Basic part :-

If $n = 1$

$p(1) = 1 < 2^1 = 1 < 2$

$\therefore p(1)$ is true.

Inductive part :-

If $n = k$

$p(k) = k < 2^k$

Now

$p(k+1) = (k+1) < 2^k \cdot 2$

$= (k+1) < 2^{k+1}$

$\therefore p(k) \to p(k+1)$ is true

7. Use M.I to P.T Sum of the 1^{st} n odd +ve integers is n^2

Basic part :-

If $n=1$

$p(1) = (1)^2 = 1$

$\therefore p(1)$ is true.

Inductive part :-

$p(k) = 1+3+5+ \dots + (2k-1) = k^2$

Now

$p(k+1) = 1+3+5+ \dots + (2k-1) + [2(k+1)-1] = k^2 + [2(k+1)-1]$

$= k^2 + 2k + 2 - 1$

$= k^2 + 2k + 1$

$= k^2+k+k+1$

$= k(k+1)+1(k+1)$

$= (k+1)(k+1) = (k+1)^2$

$\therefore p(k) \rightarrow p(k+1)$ is true.

8. use M.I to P.T $a+ar+ar^2+\dots+ar^n = \frac{ar^{n+1}-a}{a-1}$ if is a +ve integer. $\sum_{i=0}^{n} ar_i =$

Basic part:-

If $n=0$

$p(0) = \frac{ar^{0+1}-a}{r-1}$

$= \frac{ar-a}{r-1} \Rightarrow \frac{ar-a}{r-1} \Rightarrow \frac{a(r-1)}{\cancel{r-1}} = a$

$p(0) = a$

$\therefore p(0)$ is true.

Inductive part:-

$p(k) = a+ar+ar^2+\dots+ar^k = \frac{ar^{k+1}-a}{r-1}$

Now

$p(k+1) = a+ar+ar^2+\dots+ar^k+ar^{k+1} = \frac{ar^{k+1}-a}{r-1} + ar^{k+1}$

$= \frac{ar^{k+1}-a+ar^{k+1}(r-1)}{r-1}$

$= \frac{ar^{k+1}-a+ar^{k+1}r-ar^{k+1}}{r-1} \Rightarrow \frac{ar^{k+2}-a}{r-1}$

$= \frac{ar^{(k+1)+1}-a}{r-1}$

$\therefore p(k) \rightarrow p(k+1)$ is true

Recursively define functions:-

we use 2 steps to define a function with non-ve integers as its domains

Basic Steps:-

Specified the value of the function at 0.

Recursive step:-

Give a rule for finding its values and integer from its value at the smaller integer such a definition is called recursive function.

1. Suppose that 'f' is define recursively by $f(0)=3$ and $f(n+1)=2f(n)+3$ find $f(1), f(2), f(3), f(4)$.

G.T:-

$$f(0)=3 \text{ and } f(n+1)=2f(n)+3.$$

Basic step:-

$f(0)=3.$

Recursive step:-

$f(n+1)=2f(n)+3.$

If $n=1$

$f(0+1) = 2f(0)+3$

$= 2(2)+3.$

$f(1)=9$

If $n=1$

$f(1+1) = 2f(1)+3$

$= 2(9)+3$

$f(2) = 21$

If $n=2$

$f(2+1) = 2f(2)+3$

$= 2(21)+3$

$f(3) = 45$

If $n=3$

$f(3+1) = 2f(3)+3.$

$= 2(45)+3$

$f(4). = 93$

2. Give an inductive definition of the factorial $f(n)=n!$

G.T:-

$f(n)=n!.$

Basic step:-

If $n=0$

$f(0)=0!=1.$

Recursive step:-

$f(n)=n!$

Now $f(n+1)=(n+1)!$

$=(n+1)(n+1-1)!.$

$=(n+1)n!.$

3. In a recursive definition of a^n where 'a' is a non-zero real no. and n is a non-ve integer.

G.T :-

$f(n) = a^n$

Basic step :-

If $n = 0$

$f(0) = a^0 = 1.$

Recursive step :-

$f(n) = a^n$

$f(n+1) = a^{n+1} = a^n(a).$

4. Give a recursive definition $\sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \dots + a_n.$

G.T :- $\sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \dots + a_n.$

Basic step :-

If $k = 0$.

$\sum_{k=0}^{n} a_0 = a_0.$

Recursive step :-

$\sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \dots + a_n$

$\sum_{k=0}^{n+1} a_k = a_0 + a_1 + a_2 + \dots + a_n + a_{n+1}.$

5. Give a recursive definition of the function $f(n) = a_n$ where $a_n = 5^n$ find $f(1), f(2), f(3)$.

G.T :- $f(n) = a_n$ where $a_n = 5^n$.

~~If n=0~~ $\longrightarrow$ Basic step :- If $n = 0$.

$f(0) = 5^0 = 1$

Recursive step :-

$f(n) = 5^n$

If $n = 1$

$f(1) = 5^1 = 5$

If $n = 2$

$f(2) = 5^2 = 25$

If $n = 3$

$f(3) = 5^3 = 125$

Ackermann's Function:-

The famous example of a recursive function is the Ackermann's function. It may be as follows.

$$A(m,n) = \begin{cases} 2n & \text{if } m=0 \\ 0 & \text{if } m\geq 1 \ \& \ n=0 \\ 2 & \text{if } m\geq 1 \ \& \ n=1 \\ A[m-1, A(m,n-1)] & \text{if } m\geq 1 \ \& \ n\geq 2 \end{cases}$$

1. Find i) $A(1,0)$ (ii) $A(0,1)$ (iii) $A(1,1)$ (iv) $A(2,2)$.

i) $A(1,0) = 0$.

ii) $A(0,1) = 2n = 2(1) = 2$

iii) $A(1,1) = 2$

iv) $A(2,2) = A[m-1, A(m,n-1)] \Rightarrow A[2-1, A(2,2-1)]$.

$\Rightarrow A(1, A(2,1)]$.

$\Rightarrow A[1, 2]$ $\quad$ < $A(2,1)=2$.

$\Rightarrow A[m-1, A(m,n-1)]$

$\Rightarrow A[1-1, A(1,2-1)]$

$\Rightarrow A[0, A(1,1)]$.

$\Rightarrow A[0,2]$.

$\Rightarrow 2n \Rightarrow 2(2) \Rightarrow 4$.

Recursive define sets:-

The recursive definition of sets have 2 parts they are basic step and recursive step.

In the basic step an initial collection of elements, the recursive step values for forming new elements in the sets.

1. Consider the subsets of the integers define by 3.

Basic step:-

let $3 \in S$.

Recursive step:-

let $x \in S$ & $y \in S$.

Now

$3 \in S$ & $3 \in S$ = $3+3 \in S$ = $6 \in S$.

$3 \in S$ & $6 \in S$ = $3+6 \in S$ = $9 \in S$.

$3 \in S$ & $9 \in S$ $= 3+9 \in S$ $= 12 \in S$.

$3 \in S$ & $12 \in S$ $= 3+12 \in S$ $= 15 \in S$.

$\therefore$ $S = \{3, 6, 9, 12, 15\}$ is +ve integers and multiple by 3.

2. Consider the subsets of the integers defined by 2.

Basic step :-

let $2 \in S$

Recursive step:-

let $x \in S$ & $y \in S$

NOW

$2 \in S$ & $2 \in S$ $\Rightarrow 2+2 \in S$ $\Rightarrow 4 \in S$.

$2 \in S$ & $4 \in S$ $\Rightarrow 2+4 \in S$ $\Rightarrow 6 \in S$.

$2 \in S$ & $6 \in S$ $\Rightarrow 2+6 \in S$ $\Rightarrow 8 \in S$

$2 \in S$ & $8 \in S$ $\Rightarrow 2+8 \in S$ $\Rightarrow 10 \in S$

$\therefore$ $S = \{2, 4, 6, 8, 10\}$ is +ve integer and multiple by 2.

Structural Induction:-

A proof by structural Induction consists of two parts. they are basic step & Recursive step.

Basic step :-

S.T the results holds all the elements specified by the basic step of recursive definition to be in the step.

Recursive step:-

S.T if the statement is true, new elements in the recursive step of the definition the results hold for these new elements.

1. use structural Induction to P.T

$L(xy) = l(x) + l(y)$ where $x \in \epsilon^*$ is the set of string over the alphabet ε.

Let $p(y)$ is a statement.

$l(xy) = l(x) + l(y)$ where $x \in \varepsilon^*$.

NOW,

$p(y) : l(xy) = l(x) + l(y)$.

Basic step :-

let $p(y)$ is true. if $n=0$

$$p(y): l(xy) = l(x)+l(y)$$
$$= l(x)+0$$
$$= l(x)$$
$$= x \in \Sigma^*$$

Inductive step:-

$$p(ya) = l(xya) = l(x)+l(ya).$$
$$= l(x)+l(y)+l(a).$$
$$= l(xy)+l(a).$$

Strong Induction & well- ordering:-

In this section we will introduce another form of M.I is called S.I . S.I which can offer use we cannot easily prove a result using M.I, The basic step of proof by S.I is the same as a proof of same result using M.I. However the Inductive step in these two proof methods are different, in a proof by mathematical Inductive step shows that with hypothesis $p(k)$ is true then $p(k+1)$ is also true. in a proof by structural Inductive steps shows that we assume the hypothesis $p(k)$ is true. for $n=k$ values then $p(k+1)$ is also true.

***

1) the sequence defined by $a_1=1$, $a_2=8$ and $a_n=a_n+2a_{n-2}$ for $n\geq 3$ then $a_n = 3\cdot 2^{n-1}+2(-1)^n \forall x\in N$.

Let $p(n): a_n = a_{n-1}+2a_{n-2}$ for $n\geq 3$ — ① .

Also $p(n): a_n = 3\cdot 2^{n-1}+2(-1)^n \forall x\in N$ — ②

Basic step:-

If $n=1$

$$p(1): a_1 = 3\cdot 2^{1-1}+2(-1).$$

$\Rightarrow 3(1) - 2 = 1$ <by eqn ②.

$\therefore a_1 = 1$

If $n = 2$

$p(2): a_2 = 3 \cdot 2^{2-1} + 2(-1)^2$

$= 3(2) + 2$

$a_2 = 8.$

$p(n)$ is true for $n = 1, 2.$

Inductive part :-

We assume that $p(n)$ is true for $n = k$ values.

$p(k) = a_k = a_{k-1} + 2a_{k-2}.$

Now $p(k+1) : a_{k+1} = a_{k+1-1} + 2a_{k+1-2}.$

$a_{k+1} = a_k + 2a_{k-1}$

$a_{k+1} = [3 \cdot 2^{k-1} + 2(-1)^k] + 2[3(2)^{(k-1)-1} + 2(-1)^{k-1}]$

<by eqn ② $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$
$n = k, n = k-1$>

$= 3 \cdot 2^{k-1} + 2(-1)^k + 2[3 \cdot 2^{k-2} + 2(-1)^{k-1}]$

$= 3 \cdot 2^{k-1} + 2(-1)^k + 2 \cdot 3 \cdot 2^{k-2} + 2 \cdot 2(-1)^{k-1}.$

$= 3 \cdot 2^{k-1} + 2(-1)^k + 3 \cdot 2^{k-2+1} + 4(-1)^k \cdot (-1)^{-1}.$

$= 3 \cdot 2^{k-1} + 2(-1)^k + 3 \cdot 2^{k-1} - 4(-1)^k.$

$= 6 \cdot 2^{k-1} - 2(-1)^k.$

$= 3 \cdot 2 \cdot 2^{k-1} + 2(-1)^k(-1)$

$= 3 \cdot 2^{(k+1)-1} + 2(-1)^{k+1}$

$\therefore p(k+1)$ is true.

2) The Fibonacci sequence is defined by $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$ $F_1 = F_2 = 1$. S.T $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

G.T :-

$p(n): F_{n+2} = F_{n+1} = F_n$ for $n \geq 0$ — ①

$F_1 = F_2 = 1$

$p(n): F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$ — ②

Basic step :- If $n=1$

$F_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right]$

$= \frac{1}{\sqrt{5}}\left[\cancel{\frac{1}{2}} + \frac{\sqrt{5}}{2} - \cancel{\frac{1}{2}} + \frac{\sqrt{5}}{2}\right] \Rightarrow \frac{1}{\cancel{\sqrt{5}}}\left[\frac{2\cancel{\sqrt{5}}}{2}\right]$

$F_1 = 1$.

If $n=2$

$F_2 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right]$

$= \frac{1}{\sqrt{5}}\left[\frac{1^2+(\sqrt{5})^2+2\sqrt{5}}{4} - \frac{1^2+(\sqrt{5})^2-2\sqrt{5}}{4}\right]$

$= \frac{1}{\sqrt{5}}\left[\frac{\cancel{6}+2\sqrt{5}-\cancel{6}+2\sqrt{5}}{4}\right] \Rightarrow \frac{1}{\sqrt{5}}\left[\frac{\cancel{4}\sqrt{5}}{\cancel{4}}\right] \Rightarrow F_2 = 1$

Inductive step:-

we assume that $p(n)$ is true for $n=k$ values

$p(k): F_{k+2} = F_{k+1} + F_k$ ⟨by ①

$P(k+1): F_{(k+1)+2} = F_{(k+1)+1} + F_{k+1}$

$F_{k+3} = F_{k+2} + F_{k+1}$

$F_{k+1} = F_{k+3} - F_{k+2}$

$F_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+3} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+3}\right] - \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right]$

$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+3} - \left(\frac{1+\sqrt{5}}{2}\right)^{k+2}\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1-\sqrt{5}}{2}\right)^{k+3} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right]$ ⟨by ② $n = k+3, k+2$

Now consider 1st part of F_{k+1}

$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+3} - \left(\frac{1+\sqrt{5}}{2}\right)^{k+2}\right] \Rightarrow \frac{1}{\sqrt{5}}\left[a^{k+3} - a^{k+2}\right]$

⟨put $a = \frac{1+\sqrt{5}}{2}$

$= \frac{1}{\sqrt{5}}\left[a^k \cdot a^3 - a^k \cdot a^2\right] \Rightarrow \frac{1}{\sqrt{5}} a^k \cdot a^2[a-1] \Rightarrow \frac{1}{\sqrt{5}} a^k \cdot a \cdot a[a-1]$

$$= \frac{1}{\sqrt{5}} a^{k+1} \cdot a [a-1]$$

$$= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} \left(\frac{1+\sqrt{5}}{2}\right) \left[\frac{1+\sqrt{5}}{2} - 1\right]$$

$$= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} \left(\frac{1+\sqrt{5}}{2}\right) \left(\frac{1+\sqrt{5}-2}{2}\right)$$

$$= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} \left(\frac{\sqrt{5}+1}{2}\right) \left(\frac{\sqrt{5}-1}{2}\right)$$

$$= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} \left(\frac{(\sqrt{5})^2 - (1)^2}{4}\right) \Rightarrow \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} \frac{\cancel{4}}{\cancel{4}}$$

$$\Rightarrow \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{k+1}.$$

Again consider 2^{nd} part of F_{k+1}.

$$= \frac{1}{\sqrt{5}} \left[\left(\frac{1-\sqrt{5}}{2}\right)^{k+3} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right] \Rightarrow \frac{1}{\sqrt{5}} [b^{k+3} - b^{k+2}]$$

$$= \frac{1}{\sqrt{5}} [b^k b^3 - b^k b^2] \Rightarrow \frac{1}{\sqrt{5}} b^k b^2 [b-1] \quad \left\langle \text{put } b = \frac{1-\sqrt{5}}{2}\right.$$

$$\Rightarrow \frac{1}{\sqrt{5}} b^k \cdot b \cdot b [b-1] \Rightarrow \frac{1}{\sqrt{5}} b^{k+1} \cdot b [b-1]$$

$$\Rightarrow \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^{k+1} \left(\frac{1-\sqrt{5}}{2}\right) \left[\frac{1-\sqrt{5}}{2} - 1\right]$$

$$\Rightarrow \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^{k+1} \left(\frac{1-\sqrt{5}}{2}\right) \left(\frac{1-\sqrt{5}-2}{2}\right)$$

$$\Rightarrow \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^{k+1} \left(\frac{1-\sqrt{5}}{2}\right) \left(\frac{-1-\sqrt{5}}{2}\right)$$

$$\Rightarrow \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^{k+1} \left(\frac{1-\sqrt{5}}{2}\right) \left[-\left(\frac{1+\sqrt{5}}{2}\right)\right]$$

$$\Rightarrow -\frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^{k+1} \frac{(1)^2 - (\sqrt{5})^2}{4}$$

$$\Rightarrow + \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^{k+1} \left(\cancel{+} \frac{\cancel{4}}{\cancel{4}}\right)$$

$$\Rightarrow \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}$$

$$F_{k+1} = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}.$$

$$F_{k+1} = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right].$$

UNIT - 3
Basic of Counting.

Basic of counting :-

There are two basic rules of counting

* Sum rule.
* Product rule.

Sum rule :-

Suppose two tasks t_1 & t_2 are to be performed if the task 't_1' can be performed 'm' different ways and the task 't_2' can be performed 'n' different ways. If the two tasks cannot be performed simultaneously then one of the two tasks 't_1' or 't_2' can be perform in '$m+n$' ways.

Task (1)	Task (2)
t_1	t_2
m	n
$m+n$.	

Generally if $t_1, t_2 \ldots t_k$ are 'k' tasks such that two of these tasks can be perform at the same tym and if the task 't_i' can be performed in 'n_i' different ways then one of the 'k' tasks 't_1 or t_2 or t_3 or ... t_k' can be perform in '$n_1+n_2+n_3+\ldots+n_k$'.

1. Suppose there are 16 boys & 18 girls in a class, how many ways to select one of these students either a boy or girl as a class representative.

G.T :-

No. of boys = 16
No. of girls = 18

Task - 1

No. of ways to select a boy as a class representative $m = 16$.

Task - 2

No. of ways to select a girl as a class representative $n = 18$.

No. of ways to select either a boy or a girl as a class representative $= m+n$

$= 16+18$

$= 34$ ways.

2. Suppose a hostel library has 12 books on maths, 10 books on physics, 16 books on computer science and 11 books on electronics. How many ways to a student wishes to choose are of these books for study.

G.T :-

No. of maths books = 12

No. of physics books = 10.

No. of C.S books = 16.

No. of electronics books = 11.

Task 1:-

No. of ways to select maths $= 12 = n_1$

Task 2:-

No. of ways to select physics $= 10. = n_2$

Task 3:-

No. of ways to select C.S $= 16. = n_3$

Task 4:-

No. of ways to select electronics $= 11. = n_4$.

No. of ways to select either maths, physics, C.S, electronics

$= n_1 + n_2 + n_3 + n_4$

$= 12+10+16+11$

$= 49$ ways.

3) A student can choose a computer project from of the three list containing 22, 17, 13 projects resp. how many possible projects are there to choose from.

Task 1:-

A student can chaoge a computer project from 1^{st} list

$n_1 = 22$

Task 2:-

A student can choose a computer project from 2^{nd} list

$n_2 = 17$.

Task 3:-

A student can choose a computer project from 3^{rd} list

$n_3 = 13$.

No. of ways to select a computer project either 1^{st}, 2^{nd} & 3^{rd}

$n_1 + n_2 + n_3$

$= 22 + 17 + 13 = 52$ ways.

4) Find out how many no. of way for selecting prime number less than 10 and even numbers less than 10.

No. of ways for selecting prime number less than 10.

$(m) = \{2, 3, 5, 7\} = 4$

No. of ways for selecting even number less than 10

$(n) = \{2, 4, 6, 8\} = 4$

No. of ways for selecting either prime (or) even numbers less than 10 $= m+n = 4 + 4 = 8$

$8-1$ (2 is repeated both) $= 7$ ways.

Product rule:-

Suppose that two tasks t_1 & t_2 are to be performed one after the other if t_1 can be performed in n_1 different ways for each of this ways t_2 can be performed in n_1 different ways then both of the tasks can be performed in $m_1 \times n_1$ different ways

Task (1) Task (2)

T_1 T_2

m_1 n_1

$m_1 \times n_1$.

Generally if $t_1, t_2 \ldots t_k$ are k tasks such that two of these tasks can be performed at the same tym. If the tasks 't_i' can be performed in 'n_i' different ways then one of the 'k' tasks t_1 (or) t_2 (or) t_3 (or) ... t_k can be performed in $n_1 \times n_2 \times n_3 \times \ldots n_k$.

1) Suppose a person has 5 shirts and 7 ties, how many ways a person can choose a shirt and tie.

A person has 5 shirts $(m_1) = 5$

A person has 7 ties $(n_1) = 7$

No. of ways for selecting a person has 5 shirts and 7 ties = $m_1 \times n_1 = 5 \times 7 = 35$ ways.

2) There are 30 married couples in a party. Find the no. of ways for choosing one men and one women from the party, such that two ~~are~~ ~~not~~ (or) not married.

G.T:-

30 couples in a party $= 30 \times 2 = 60$

No. of ways to choosing a man and women in a party such that two (or) not married to each other $30 \times 29 = 870$ ways.

3) There are 32 micro computers in a computer centre each micro computer has 24 ports, how many different ports to a micro computer in the centre are there

Total no. of micro computers in a computer center $(m_1) = 32$

Each micro computers has 24 ports $(n_1) = 24$

Total no. of micro computers ports in a computer centre =

$$m_1 \times n_1$$
$$= 32 \times 24$$
$$= 768 \text{ ways.}$$

4) How many different bit strings of the length 7 are there.

The bit $\{0,1\} = 2$

The bit string length $= 7$

The no. of way for bit string $= 2^7 = 128$.

5) The chairs of an auditorim are to be label with a letter and a +ve integer not exceeding 100. wt is the largest no. of chairs that can be label differently.

The no. of alphabets $(m_1) = 26$.

The +ve integer not exceeding $(n_1) = 100$.

The largest no. of chairs $= m_1 \times n_1$

$= 26 \times 100$

$= 2600$ ways.

Pigeon-hole principle:-

If 'n' pigeons accomidated in 'm' pigeon holes and $n > m$ atleast one pigeon hole will contain two or more pigeons.

Generalised pigeon hole principle:-

If 'n' pigeons are accomidated in 'm' pigeon holes are $n > m$ then one of the pigeon holes must contain $\left[\frac{n-1}{m}\right]+1$. pigeons.

1. Suppose there are 26 students in 7 cars to transport them S.T atleast one car must have 4 or more passengers.

The no. of students $(n) = 26$

The no. of cars $(m) = 7$

$n > m$

i.e., $26 > 7$

Generalized pigeon hole principle

$$\left[\frac{n-1}{m}\right]+1 = \left[\frac{26-1}{7}\right]+1 = \frac{25}{7}+1 = \frac{32}{7} = 4.57$$

$\therefore$ 5 students approximately.

2. Suppose there are 26 passengers in a 5 mini bus to transport them. S.T atleast there are one bus must have 5 (or) more passengers.

the no. of passengers $(n) = 26$
the no. of mini bus $(m) = 5$

$n > m$

i.e., $26 > 5$

Generalized pigeonhole principle.

$$\left[\frac{n-1}{m}\right]+1 = \left[\frac{26-1}{5}\right]+1 = \frac{25}{5}+1 = 6.$$

3) S.T in any grp of 30 people we can always find 5 people who were born on the same day of the week.

the no. of people $(n) = 30$.
No. of days $(m) = 7$.

$n > m$

i.e., $30 > 7$

Generalized pigeon hole principle.

$$\left[\frac{n-1}{m}\right]+1 = \left[\frac{30-1}{7}\right]+1 = \frac{29}{7}+1 = \frac{36}{7} = 5.14.$$

$\therefore$ 5 people were born on the same day in a week.

4) There are 25 boxes with 3 variety of apples each box contains only one kind of apple P.T there are atleast 9 boxes with apples of the same variety.

the no. of boxes $(n) = 25$
the no. of variety apples $(m) = 3$

$n > m$

i.e., $25 > 3$

Generalized pigeonhole principle.

$$\left[\frac{n-1}{m}\right]+1$$

$$= \left[\frac{25-1}{3}\right]+1 = \frac{24}{3}+1 = 9.$$

5) P.T if 7 natural no.s are choosen at random then the sum of three values of them is divisible by 3.

The natural no.s $(n) = 7$
The sum values $(m) = 3$.

$n > m$

i.e., $7 > 3$

Generalized pigeon hole principle.

$= \left[\frac{n-1}{m}\right]+1$

$= \left[\frac{7-1}{3}\right]+1 = \frac{6}{3}+1 = 3.$

Permutations and Combinations :-

An ordered arranged of 'r' elements of a set combining 'n' distinct elements is a 'r'- permutation (or) permutations of 'n' elements and it is denoted by $p(n,r)$ (or) $^nP_r = \frac{n!}{(n-r)!}$, $r \le n$

1. P.T the no. of permutations of 'n' things taken all at a tym is $n!$.

WKT

$$p(n,r) \text{ (or) } {}^nP_r = \frac{n!}{(n-r)!}$$

$< n=r.$

$$^nP_n = \frac{n!}{(n-n)!}$$

$$= \frac{n!}{0!} = \frac{n!}{1} = n!.$$

2. How many ways are there to select a 1^{st} prize winner, 2^{nd} prize winner and a third prize winner from 100 different people we have entered a contest.

the no. of people $n=100$.

The permutation (1^{st}, 2^{nd}, 3^{rd} prizes) $(r)=3$.

$$\therefore {}^nP_r = {}^{100}P_3$$

$$^nP_r = \frac{n!}{(n-r)!}$$

$$^{100}P_3 = \frac{100!}{(100-3)!} = \frac{100!}{97!} = \frac{100\times99\times98\times\cancel{97!}}{\cancel{97!}} = 970200 \text{ ways}$$

3. Suppose that there are 8 runners in a race the winners receive a gold medal of the 1^{st} place, 2^{nd} place finisher receives silver medal and 3^{rd} place finisher receives bronze medal, how many different ways are there to award these medals if all possible outcomes of the race and there are no ties.

No. of runners $(n)=8$.

No. of permutions (gold, silver, bronze) $(r)=3$

$$^nP_r = \frac{n!}{(n-r)!}.$$

$^8P_3 = \frac{8!}{(8-3)!}$

$= \frac{8!}{5!}$

$= 8\times7\times6\times5 = 336$ ways.

4) How many different strings of length 4 can be formed using the letters by the word PROBLEM.

The word PROBLEM has 7 letters i.e., $n=7$

the string length $(r) = 4$

$^nP_r = \frac{n!}{(n-r)!}$

$^7P_4 = \frac{7!}{(7-4)!}$.

$= \frac{7!}{3!} = \frac{7\times6\times5\times4\times\cancel{3!}}{\cancel{3!}} = 840$ ways.

5) How many words of 3 distinct letters can be formed using the letters of the word PASCAL.

The word PASCAL has 6 letters $n=6$.

the no. of 3 distinct letters $r=3$

$^nP_r = \frac{n!}{(n-r)!}$

$= \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6\times5\times4\times\cancel{3!}}{\cancel{3!}} = 120$.

Generalized permutations:-

No. of permutations of 'n' obj it is req. to find the no. of permutations that can be form a collection of 'n' objects of which n_1 is 1st type, n_2 is 2nd type.... n_k is k type then the no. of permutation 'n' obj is $\frac{n!}{n_1!\times n_2!\times\ldots\times n_k!}$

1. find the no. of permutations of the letters ENGINEERING

The word ENGINEERING has 11 letters out of which 3-E's, 3-N's, 2-I's, 2-G's, 1-R

Now taking $n_1=3$, $n_2=3$, $n_3=2$, $n_4=2$, $n_5=1$ and $n=11$

The no. of permutation of n objects

$\frac{n!}{n_1!\times n_2!\times\cdots\times n_k!}$

$$= \frac{11!}{3! \times 3! \times 2! \times 2! \times 1!} = \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times \cancel{3!}}{\cancel{3!} \times 3 \times 2 \times 2 \times 1 \times 2 \times 1 \times 1}$$

$$= 2,77,200$$

2) Find the no. of permutations of the letters i) SUCCESS

The word SUCCESS has 7 letters out of which 3-S's, 2-C's, 1-U, 1-E

Now taking $n_1 = 3$, $n_2 = 2$, $n_3 = 1$, $n_4 = 1$, $n = 7$

the no. of permutation of n obj.

$$\frac{n!}{n_1! \times n_2! \times \cdots \times n_k!}$$

$$= \frac{7!}{3! \times 2! \times 1! \times 1!} = \frac{7 \times 6 \times 5 \times 4 \times \cancel{3!}}{\cancel{3!} \times 2 \times 1 \times 1 \times 1} = 14 \times 30 = 420.$$

ii) EXCELLENT iii) ELECTRIC.

ii) The word ~~SUCCESS has~~ EXCELLENT has 9 letter out of which 3-E's, 1-X, 1-C, 2-L's, 1-N, 1-T

Now taking $n_1 = 3$, $n_2 = 1$, $n_3 = 1$, $n_4 = 2$, $n_5 = 1$, $n_6 = 1$ & $n = 9$

The no. of permutation of n obj

$$\frac{n!}{n_1! \times n_2! \times \cdots \times n_k!}$$

$$= \frac{9!}{3! \times 1! \times 1! \times 2! \times 1! \times 1!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times \cancel{3!}}{\cancel{3!} \times 1 \times 1 \times 2 \times 1 \times 1 \times 1} = \frac{362880}{2}$$

$$= \frac{60,480}{2} = 30,240.$$

iii) The word ELECTRIC has 8 letters out of which 2-E's, 1-L, 2-C's, 1-T, 1-R, 1-I

Now taking $n_1 = 2$ $n_2 = 1$ $n_3 = 2$ $n_4 = 1$ $n_5 = 1$ $n_6 = 1$ $n = 8$

the no. of permutation of n obj

$$\frac{n!}{n_1! \times n_2! \times \cdots \times n_k!} = \frac{8!}{2! \times 1! \times 2! \times 1! \times 1! \times 1!}$$

$$= \frac{8 \times 7 \times 6 \times 5 \times \cancel{4}^{2} \times 3 \times \cancel{2!}}{\cancel{2!} \times 1 \times \cancel{2} \times 1 \times 1 \times 1 \times 1}$$

$$= 10,080.$$

3) A ship must have raise 8 flags at one tym, are 2 red, 2 blue, 4 greens, how many different flags combinations can be raised at a tym.

Given

$n = 8, n_1 = 2, n_2 = 2, n_3 = 4$

No. of permutations of 'n' obj.

$$\frac{n!}{n_1! \times n_2! \times \ldots \times n_k!}$$

$$= \frac{8!}{2! \times 2! \times 4!} = \frac{\overset{2}{\cancel{8}} \times 7 \times 6 \times 5 \times \cancel{4!}}{\cancel{2} \times 1 \times \cancel{2} \times 1 \times \cancel{4!}}$$

$= 420.$

Circular permutations:-

Permutation in a circle is called "circular permutation" the no of ways are arranging in the circle is $(n-1)!$

1. How many ways are there to sit 10 boys and 10 girls around a circular table.

G.T:-

10 boys and 10 girls around a circular table

Total no. of persons $= 10 + 10 = 20$

Total no. of permutations of circular table is $(n-1)!$

$= (20-1)! = 19!$.

2. How many ways are there 3 persons sit around round table.

G.T:-

Total no. of persons $= 3$

total no. of permutations of round table is $(n-1)!$

$= (3-1)! = 2! = 2$

Combinations:-

The no. of 'r' combination of a set with 'n' elements, where 'n' is a non-ve integer and 'r' is an integer with $0 \le r \le n$ is equal to $C(n, r)$ (or) $n_{c_r} = \frac{n!}{r!(n-r)!}$

1. A grp of 30 people have been trained as astronauts to go on the 1st machine to mars. How many ways are there to select a crew of 6 people to go on this machine.

No. of people $n = 30$.

No. of permutations $r = 6$.

$$^nC_r = \frac{n!}{r!(n-r)!} \Rightarrow {}^{30}C_6 = \frac{30!}{6!(30-6)!}$$

$$\Rightarrow \frac{30!}{6!(24)!} \Rightarrow \frac{30\times29\times28\times27\times26\times25\times\cancel{24!}}{6\times5\times4\times3\times2\times1\times\cancel{24!}}$$

$$\Rightarrow 29\times63\times13\times25$$

$$= 593775$$

2. Suppose that there are 9 faculty members in the mathem-atics department and 11 in the Computer Science department, how many ways are there to select a Committee to develop a discrete mathematics course at a Scl. if the Committee is to consists of 3 faculty members from the mathematics department and 4 from the Computer Science department.

Mathematics department $(n) = 9$

No. of ways $(r) = 3$

computer science department $(n) = 11$

No. of ways $(r) = 4$

By product rule

$$^9C_3 \times {}^{11}C_4 = \frac{9!}{3!(9-3)!} = \frac{11!}{4!(11-4)!}$$

$$= \frac{9!}{3!\,6!} \times \frac{11!}{4!\,7!}$$

$$= \frac{9\times8\times7\times\cancel{6!}}{3\times2\times1\times\cancel{6!}} \times \frac{11\times10\times9\times8\times\cancel{7!}}{4\times3\times2\times1\times\cancel{7!}}$$

$$= 110\times36\times7$$

$$= 27720.$$

Bi-nomial coefficients:-

The no. of 'r' ~~co-efficients~~ Combinations from a set with 'n' elements is often is denoted by ^nc_r (or) $\binom{n}{r}$ $c(n,r)$. This number is also called a binomial co-efficient because these no.s occur as co-efficients in the ~~eqn~~ expansion of powers of binomial expansion such as

$$(x+y)^n = {}^nc_0\, x^{n-0} y^0 + {}^nc_1\, x^{n-1} y^1 + {}^nc_2\, x^{n-2} y^2 + \dots + {}^nc_r\, x^{n-r} y^r + \dots$$
$$\dots + {}^nc_n\, x^{n-n} y^n$$

(or)

$$(x+y)^n = \sum_{r=0}^{n} {}^nc_r\, x^{n-r} y^r$$ where x, y are real no.s

1) wt is the expansion of $(x+y)^4$.

Here $x = x$, $y = y$, $n = 4$.

$$\boxed{\begin{matrix} {}^nc_0 = 1 \\ {}^nc_n = 1 \end{matrix}}$$

$$(x+y)^n = {}^nc_0\, x^{n-0} y^0 + {}^nc_1\, x^{n-1} y^1 + {}^nc_2\, x^{n-2} y^2 + {}^nc_3\, x^{n-3} y^3 + \dots$$

$$(x+y)^4 = {}^4c_0\, x^{4-0} y^0 + {}^4c_1\, x^{4-1} y^1 + {}^4c_2\, x^{4-2} y^2 + {}^4c_3\, x^{4-3} y^3 +$$
$${}^4c_4\, x^{4-4} y^4.$$

$$= (1)x^4(1) + 4x^3y + 6x^2y^2 + 4x^1y^2 + (1)x^0y^4.$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

2) wt is the expansion $(4a+3d)^4$

Here $x = 4a$, $y = 3d$, $n = 4$

$$(x+y)^n = {}^nc_0\, x^{n-0} y^0 + {}^nc_1\, x^{n-1} y^1 + {}^nc_2\, x^{n-2} y^2 + {}^nc_3\, x^{n-3} y^3 + \dots$$

$$(4a+3d)^4 = {}^4c_0\, 4a^{4-0} y^0 + {}^4c_1\, (4a)^{4-1} 3d^1 + {}^4c_2\, (4a)^{4-2} 3d^2 + {}^4c_3\, (4a)^{4-3} 3d$$
$$+ {}^4c_4\, (4a)^{4-4} 3d^4.$$

$$\Rightarrow (1)(4a)^4(1) + 4(4a)^3(3d) + 6(4a)^2(3d)^2 + 4(4a)^1(3d)^3 + (1)(1)3d^4$$

$$\Rightarrow 256a^4 + 768a^3d + 864a^2d^2 + 432ad^3 + 81d^4.$$

3) wt is the expansion $(x+2)^6$.

Here $x = x$ $y = 2$ $n = 6$.

$$(x+y)^n = {}^nc_0\, x^{n-0} y^0 + {}^nc_1\, x^{n-1} y^1 + {}^nc_2\, x^{n-2} y^2 + {}^nc_3\, x^{n-3} y^3 + \dots$$

$(x+2)^6 = {}^6C_0 x^{6-0} 2^0 + {}^6C_1 x^{6-1} 2^1 + {}^6C_2 x^{6-2} 2^2 + {}^6C_3 x^{6-3} 2^3 + {}^6C_4 x^{6-4} 2^4 + {}^6C_5 x^{6-5} 2^5 + {}^6C_6 x^{6-6} 2^6.$

$= (1) x^{6-0} (1) + 6 x^{6-1} 2^1 + 15 x^{6-2} 2^2 + 20 x^{6-3} 2^3 + 15 x^{6-4} 2^4 + 6 x^{6-5} 2^5 + 1 . x^{6-6} 2^6.$

$= 1 x^6 1 + 6x^5 2 + 15 x^4 . 4 + 20 x^3 8 + 15 x^2 . 16 + 6x^1 32 + 1 x^0 64.$

$= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64.$

4) Find out the co-efficient of $x^9 y^3$ in the expansion of $(x+2y)^{12}$.

Given expansion $(x+2y)^{12}$, Here $x = x$, $y = 2y$, $n = 12$

$(x+y)^n = \sum_{r=0}^{n} {}^nC_r \, x^{n-r} y^r.$

$(x+2y)^{12} = \sum_{r=0}^{n} {}^{12}C_r \, x^{12-r} (2y)^r.$

$(x+2y)^{12} = \sum_{r=0}^{n} {}^{12}C_r \, 2^r x^{12-r} y^r$ — ①

To find coefficient of $x^9 y^3$.

Now, $x^{12-r} = x^9$	$y^r = y^3$
$12-r = 9$	$r = 3$
$r = 3$.	

Sub $r=3$ in eqn ①

$(x+2y)^{12} = \sum_{r=0}^{3} {}^{12}C_3 \, 2^3 . x^{12-3} . y^3.$

$= \sum_{r=0}^{3} {}^{12}C_3 (8) x^9 y^3.$

The coefficient of $x^9 y^3 = {}^{12}C_3 (8)$

$= \frac{12!}{3!(12-3)!} \times 8$

$= \frac{12!}{3!\,9!} \times 8 \Rightarrow \frac{\cancel{12}^{4} \times 11 \times 10 \times \cancel{9!}}{3 \times \cancel{2} \times 1 \times \cancel{9!}} \times 8$

$\Rightarrow 44 \times 40$

$\Rightarrow 1760.$

5) What is the coefficient of $x^{12} y^{13}$ in the expansion of $(2x-3y)^{25}$.

Given expansion $(2x-3y)^{25}$

Here $x=2x$, $y=3y$, $n=25$

$$(x+y)^n = \sum_{r=0}^{n} {}^{n}C_r \, x^{n-r} y^r.$$

$$(2x-3y)^{25} = \sum_{r=0}^{n} {}^{25}C_r (2x)^{25-r} (-3y)^r.$$

$$= \sum_{r=0}^{n} {}^{25}C_r (2)^{25-r} (-3)^r x^{25-r} y^r.$$

To find coefficient of $x^{12} y^{13}$.

Now

$x^{25-r} = x^{12}$	$y^r = y^{13}$
$25-r = 12$	$r=13$
$r = 13.$	

Sub $r=13$ in eqn ①

$$(2x-3y)^{25} = \sum_{r=0}^{13} {}^{25}C_{13} (2)^{25-13} (-3)^{13} x^{25-13} y^{13}.$$

$$= \sum_{r=0}^{13} {}^{25}C_{13} (2)^{12} (-3)^{13} x^{12} y^{13}.$$

The coefficient of $x^{12}y^{13} = {}^{25}C_{13} (2)^{12} (-3)^{13}.$

$$= \frac{25!}{13!(25-13)!} \cdot 2^{12} (-1)^{13} (3)^{13}.$$

$$= -\frac{25!}{13!\,12!} (2)^{12} (3)^{13}.$$

Generalized Combinations:

You can use the formula below to find out the no. of combinations when repetition is allowed

$$^{n}c_{r} = \frac{r+(n-1)!}{r!\,(n-1)!}$$

Here 'n' = total no. of elements in a set

'r' = No. of that can be selected from a set

Let suppose there are 'p' elements in a set are asked to selected on elements from this set given that each element can be selected multiples is called as a combination with repitition.

Problems:

1) There are 5 colors balls in a pool all balls are different colors in how many ways can be choose 4 pool balls.

Sol: Total no. of balls (n) = 5

No. of balls to be selected (r) = 4

$$^{n}c_{r} = \frac{(r-(n-1))!}{r!\,(n-1)!}$$

$$^{5}c_{4} = \frac{(4+5-1)!}{4!\,(5-1)!}$$

$$= \frac{8!}{4!\,4!}$$

$$= \frac{8\times7\times6\times5\times4!}{4!\times4\times3\times2\times1}$$

$$= 70$$

2) There are 8 different ice-creams flavours in the ice-cream shop in how many day can be choose 5 flavours out of this 8 flavours.

Sol: Total no. of Ice-cream flavours (n) = 8

No. of Ice-cream to be selected (r) = 5

$$^{n}c_{r} = \frac{(r+n-1)!}{r!\,(n-1)!}$$

$${}^{5}C_{4} = \frac{(5+8-1)!}{5!\,(8-1)!}$$

$$= \frac{12!}{5!\,7!}$$

$$= \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7!}{5 \times 4 \times 3 \times 2 \times 1 \times 7!}$$

$$= 11 \times 72$$

$$= 792$$

3) There are 5 color pencils in how many day can be selected three pencils

Sol: Total no. of pencils$(n) = 5$

No. of pencils to be selected $(r) = 3$

$${}^{n}C_{r} = \frac{(r+n-1)!}{r!\,(n-1)!}$$

$${}^{5}C_{3} = \frac{(3+5-1)!}{3!\,(5-1)!}$$

$$= \frac{7!}{3!\,4!}$$

$$= \frac{7 \times 6 \times 5 \times 4!}{3 \times 2 \times 1 \times 4!}$$

$$= 35$$

Generate permutations:

It is based on lexicography (Ascending order)

Problems:

1) Place this permutations of the set (1.2.3.4.5) in lexicography.

43521, 15432

45321, 23451

23514, 14532

21345, 45230

31452, 34542

Ans:

14532 31452

15432 34542

21345 45230

23451 43521

23514 45321

3) Place the permutations of the set $\{1,2,3,4,5,6\}$

Sol:	234561	231455	Sol:	156423	43256
	231456	314562		165432	435612
	165432	43256		231455	541236
	156423	654321		231456	543216
	543216	654312		234561	654312
	541236	435612		314562	654321

UNIT-IV

Advance counting technique.

Reference Relation:-

A reference relation for a sequence ~~an~~ is an eqn that express a_n in terms of one (or) more of the previous te of the sequence namely $a_n = a_0, a_1, a_2 \ldots a_{n-1}$ ∀ integers 'n' with $n \geq n_0$.

where 'n_0' is the non-negative integer this sequence is called a solution of a recurrence relation.

If it terms satisfy the recurse relation terms then

$$a_n = a_{n-1} + [\text{common difference}]$$

(or)

$$a_n = a_{n-1} * [\text{common difference}]$$

1) Find the recurrence relation for the given series 5, 8, 11, 14, 17?.

Given series (S) = 5, 8, 11, 14, 17, ?.

Let $a_0 = 5$

Recurrence relation $a_n = a_{n-1} + [\text{common difference}]$

$$a_n = a_{n-1} + 3$$

put n=1

$a_1 = a_{1-1} + 3$

$= a_0 + 3$

$= 5+3$

$a_1 = 8$

put n=2

$a_2 = a_{2-1} + 3$

$= a_1 + 3$

$= 8+3$

$a_2 = 11$

put n=3

$a_3 = a_{3-1} + 3$

$= a_2 + 3$

$= 11+3$

$a_3 = 14$

put n=4

$a_4 = a_{4-1} + 3$

$= a_3 + 3$

$= 14+3$

$a_4 = 17$

put n=5

$a_5 = a_{5-1} + 3$

$= a_4 + 3$

$= 17+3$

$a_5 = 20$

∴ The sequence is 5, 8, 11, 14, 17, 20

2) find the sequence generated by the recurrence relation.

$a_n = 2a_{n-1}$ with $a_1 = 4$.

Given recurrence relation $a_n = 2a_{n-1}$ with $a_1 = 4$

put $n=2$

$a_2 = 2a_{2-1}$

$= 2a_1$

$= 2(4)$

$a_2 = 8$

put $n=3$

$a_3 = 2a_{3-1}$

$= 2a_2$

$= 2(8)$

$a_3 = 16$

put $n=4$

$a_4 = 2a_{4-1}$

$= 2a_3$

$= 2(16)$

$a_4 = 32$

put $n=5$

$a_5 = 2a_{5-1}$

$= 2a_4$

$= 2(32)$

$a_5 = 64$

$\therefore$ the sequence is 8, 16, 32, 64, ...

3) find the sequence generated by the recurrence relation

2, 6, 18, 54, 162, ?.

Given series (s) : 2, 6, 18, 54, 162, ?

Let $a_0 = 2$

Recurrence Relation $a_n = a_{n-1} \times$ [Common difference]

$a_n = a_{n-1} \times 3$

put $n=1$

$a_1 = a_{1-1} \times 3$

$= a_0 \times 3$

$= 2 \times 3$

$a_1 = 6$

put $n=2$

$a_2 = a_{2-1} \times 3$

$= a_1 \times 3$

$= 6 \times 3$

$a_2 = 18$

put $n=3$

$a_3 = a_{3-1} \times 3$

$= a_2 \times 3$

$= 18 \times 3$

$a_3 = 54$

put $n=4$

$a_4 = a_{4-1} \times 3$

$= a_3 \times 3$

$= 54 \times 3$

$a_4 = 162$

put $n=5$

$a_5 = a_{5-1} \times 3$

$= a_4 \times 3$

$= 162 \times 3$

$a_5 = 486$

$\therefore$ the sequence is 2, 6, 8, 54, 162, 486.

4) Find the 1st 5 terms on the sequence defined by

$a_n = 6a_{n-1}$ with $a_0 = 2$

Given recurrence.

$a_n = 6a_{n-1}$ with $a_0 = 2$

put $n=1$

$a_1 = 6a_{1-1}$

$= 6a_0$

$= 6(2)$

$a_1 = 12$

put $n=2$

$a_2 = 6a_{2-1}$

$= 6a_1$

$= 6(12)$

$a_2 = 72$

put $n=3$

$a_3 = 6a_{3-1}$

$= 6a_2$

$= 6(72)$

$a_3 = 432$

put $n=4$

$a_4 = 6a_{4-1}$

$= 6a_3$

$= 6(432)$

$a_4 = 2,592$

put $n=5$

$a_5 = 6a_{5-1}$

$= 6a_4$

$= 6(2592)$

$a_5 = 15,552$

$\therefore$ The 1st 5 terms are 12, 72, 432, 2592, 15552.

5) A sequence that satisfy the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4 \ldots$ and suppose that $a_0 = 3$, $a_1 = 5$. what are the values of a_2 & a_3.

put $n=2$

$a_2 = a_{2-1} - a_{2-2}$

$= a_1 - a_0$

$= 5 - 3$

$a_2 = 2$

put $n=3$

$a_3 = a_{3-1} - a_{3-2}$

$= a_2 - a_1$

$= 2 - 5$

$a_3 = -3$

6) find the first five terms of the sequence defined by $a_n = na_{n-1} + n^2 a_{n-2}$, $a_0 = 1$, $a_1 = 1$.

put $n=2$

$a_2 = 2 \cdot a_{2-1} + 2^2 a_{2-2}$

$= 2a_1 + 4a_0$

$= 2a_1 + 4(1)$

$= 2a_1 + 4$

$= 2(1) + 4$

$a_2 = 6$

put $n=3$

$a_3 = 3a_{3-1} + 3^2 a_{3-2}$

$= 3a_2 + 9a_1$

$= 3(6) + 9(1)$

$= 18 + 9$

$a_3 = 27$

put $n=4$

$$a_4 = 4a_{4-1} + 4^2 a_{4-2}$$
$$= 4a_3 + 16a_2$$
$$= 4(27) + 16(6)$$
$$a_4 = 204$$

put $n=5$

$$a_5 = 5a_{5-1} + 5^2 a_{5-2}$$
$$= 5a_4 + 25a_3$$
$$= 5(204) + 25(27)$$
$$a_5 = 1675$$

7) find the 1^{st} 4 terms of sequence defined by

$a_n = 3a_{n-1}^2$, $a_0 = 1$.

put $n=1$

$$a_1 = 3a_{1-1}^2$$
$$= 3a_0^2$$
$$= 3(1)^2$$
$$a_1 = 3$$

put $n=2$

$$a_2 = (3a_{2-1}^2)$$
$$= 3a_1^2$$
$$= 3(3)^2$$
$$a_2 = 27$$

put $n=3$

$$a_3 = 3a_{3-1}^2$$
$$= 3a_2^2$$
$$= 3(27)^2$$
$$a_3 = 2187$$

8) find the next six terms of sequence is defined by

$a_n = a_{n-1} - a_{n-2} + a_{n+3}$, $a_0 = 1$, $a_1 = 1$, $a_2 = 2$

put $n=3$

$$a_3 = a_{3-1} - a_{3-2} + a_{3-3}$$
$$a_3 = a_2 - a_1 + a_0.$$
$$a_3 = 2 - \cancel{1} + \cancel{1}$$
$$a_3 = 2$$

put $n=4$

$$a_4 = a_{4-1} - a_{4-2} + a_{4-3}.$$
$$= a_3 - a_2 + a_1$$
$$= \cancel{2} - \cancel{2} + 1$$
$$a_4 = 1$$

put $n=5$

$$a_5 = a_{5-1} - a_{5-2} + a_{5-3}$$
$$= a_4 - a_3 + a_2$$
$$= 1 - \cancel{2} + \cancel{2}$$
$$a_5 = 1$$

put $n=6$

$$a_6 = a_{6-1} - a_{6-2} + a_{6-3}$$
$$= a_5 - a_4 + a_3$$
$$= \cancel{1} - \cancel{1} + 2$$
$$a_6 = 2$$

put $n=7$

$$a_7 = a_{7-1} - a_{7-2} + a_{7-3}$$
$$= a_6 - a_5 + a_4$$
$$= 2 - \cancel{1} + \cancel{1}$$
$$a_7 = 2$$

put $n=8$

$$a_8 = a_{8-1} - a_{8-2} + a_{8-3}$$
$$= a_7 - a_6 + a_5$$
$$= \cancel{2} - \cancel{2} + 1$$
$$a_8 = 1$$

9) Find the first six terms of the sequence defined by $a_n = -2a_{n-1}$, $a_0 = -1$.

put $n=1$

$a_1 = -2a_{1-1}$

$= -2a_0$

$= -2(-1)$

$a_1 = 2$

put $n=2$

$a_2 = -2a_{2-1}$

$= -2a_1$

$= -2(2)$

$a_2 = -4$

put $n=3$

$a_3 = -2a_{3-1}$

$= -2a_2$

$= -2(-4)$

$a_3 = 8$

put $n=4$

$a_4 = -2a_{4-1}$

$= -2a_3$

$= -2(8)$

$a_4 = -16$

put $n=5$

$a_5 = -2a_{5-1}$

$= -2a_4$

$= -2(-16)$

$a_5 = 32$

First order linear (or) homogeneous relation :-

Linear recurrence relation of first order constant co-efficient is $a_n = ca_{n-1} + f(n)$ where c is a constant and $f(n)$ is a known factor such relation is called a linear recurrence relation of first order with constant co-efficient.

In eqn call if $f(n)=0$ then the relation is called first order homogeneous recurrence relation. otherwise non-homogeneous relation.

1) If $a_n = [ca_{n-1} + f(n)]$ proof can be solved in substitute n by n+1 in eqn.

Given :-

$a_n = ca_{n-1} + f(n)) \longrightarrow ①$

Sub 'n' by n+1

$a_{n+1} = a_{n+1-1} + f(n+1)$

$a_{n+1} = [ca_n + f(n+1)] \longrightarrow ②$

If $n=0$

$a_{0+1} = ca_0 + f(0+1))$

$a_1 = [ca_0 + f(1)]$

If $n=1$

$a_{1+1} = ca_1 + f(1+1))$

$a_2 = c[ca_0 + f(1)] + f(2)]$

$a_2 = c^2 a_0 + c f(1) + f(2)$

If $n=2$

$a_{2+1} = c a_2 + f(2+1)$

$a_3 = c[c^2 a_0 + c f(1) + f(2)] + f(3)$.

$a_3 = c^3 a_0 + c^2 f(1) + c f(2) + f(3)$

$\vdots$

$a_n = c^n a_0 + c^{n-1} f(1) + c^{n-2} f(2) + \cdots + c f(n-1) + f(n)$.

$a_n = c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k)$.

$\therefore$ this is general solution of recurrence relation eqn ② which is equivalent to eqn ① If $f(k)=0$ in eqn ⑤ then the recurrence relation is homogeneous eqn.

NOTE :-

$a_n = c^n a_0$ for $n \geq 1$ this is a solution of first order linear (or) homogeneous recurrence relation.

2) Solve the recurrence relation of first order with constant coefficient is $a_{n+1} - 4a_n$ for $n \geq 0$ with $a_0 = 3$.

$a_{n+1} = 4a_n$ —① for $n \geq 0$, $a_0 = 3$

1st order

By linear (or) homogeneous recurrence relation is

$a_n = c^n a_0$ for $n \geq 1$ —②.

Now taking $a_n = c^n$ — ③

sub n by $n+1$.

$a_{n+1} = c^{n+1}$

$4a_n = c^{n+1}$ (by ①

$4c^n = c^n \cdot c$ (by ③

$\boxed{c=4}$

from ②

$a_n = (4)^n (3)$ for $n \geq 1$.

3) Solve the recurrence relation $a_n = 7a_{n-1}$ for $n \geq 1$, $a_2 = 98$

$a_n = 7a_{n-1}$ —① for $n \geq 1$ & $a_2 = 98$.

Sub n by $n+1$ in eqn ①.

$a_{n+1} = 7a_{n+1-1}$

$a_{n+1} = 7a_n$ —— ②

1^{st} order.

By linear (or) homogeneous recurrence relation is

$a_n = c^n a_0$ for $n \geq 1$ —— ③.

Now taking $a_n = c^n$ —— ④

Sub n by $n+1$

$a_{n+1} = c^{n+1}$

$7a_n = c^n . c$ <by ②.

$7c^n = c^n . c$ <by ①.

$\boxed{c = 7}$

from ③

$a_n = 7^n . a_0$.

If $n = 2$

$a_2 = (7)^2 a_0$

$98 = 49a_0$

$\boxed{a_0 = 2}$

$\therefore a_n = c^n . a_0$.

$a_n = 7^n . 2$

4) Solve the following first order linear (or) homogeneous recurrence relation $4a_n - 5a_{n-1} = 0$ for $n \geq 1$ & $a_0 = 1$.

Given:-

$4a_n - 5a_{n-1} = 0$.

Sub 'n' by $n+1$ in the above eqn.

$4a_{n+1} - 5a_{n+1-1} = 0$.

$4a_{n+1} - 5a_n = 0$

$4a_{n+1} = 5a_n$

$a_{n+1} = \frac{5}{4} a_n$ —— ①

By first order linear (or) homogeneous R.R is

$a_n = c^n a_0$ —— ②

Now taking $a_n = c^n$ —— ③

Sub. n by n+1 in eq ③

$a_{n+1} = c^{n+1}$ <by ①

$\frac{5}{4} a_n = c^n \cdot c$

$\frac{5}{4} c^n = c^n \cdot c$ <by ③

$\boxed{c = \frac{5}{4}}$

from ②

$a_n = \left(\frac{5}{4}\right)^n a_0$

$a_n = \left(\frac{5}{4}\right)^n (1)$

5) Solve the recurrence relation $3a_{n+1} - 4a_n = 0$ for $n \geq 0$ & $a_1 = 5$

Given:-

$3a_{n+1} - 4a_n = 0$

$3a_{n+1} = 4a_n$

$a_{n+1} = \frac{4}{3} a_n$ —— ①

By 1st order linear (or) HRR.

$a_n = c^n a_0$ —— ②

Now taking $a_n = c^n$ —— ③

Sub n by n+1 in eqn ③

$a_{n+1} = c^{n+1}$ <by ①

$\frac{4}{3} a_n = c^n \cdot c \Rightarrow \frac{4}{3} \cancel{c^n} = \cancel{c^n} \cdot c$

$c = \frac{4}{3}$

from ②

$a_n = \left(\frac{4}{3}\right)^n a_n$

put n=1

$a_1 = \left(\frac{4}{3}\right)^1 a_0$

$5 = \left(\frac{4}{3}\right) a_0$

$\frac{15}{4} = a_0$

$\boxed{a_0 = \frac{15}{4}}$

Second order linear (or) homogeneous recurrence relation:-

The second order linear (or) homogeneous R.R with constant co-efficient as the following form.

theorem:-

$c^n a_n + c^{n-1} a_{n-1} + c^{n-2} a_{n-2} = 0$ for $n \geq 2$.

Proof:-

Sub $a_n = k^n$ in eqn ①

$c^n k^n + c^{n-1} k^{n-1} + c^{n-2} k^{n-2} = 0$.

for $n = 2$

$c^2 k^2 + c^{2-1} k^{2-1} + c^{2-2} k^{2-2} = 0$.

$c^2 k^2 + ck + 1 = 0$.

case (i):-

If the roots are real & distinct.

$a_n = A(k_1)^n + B(k_2)^n$.

case (ii):-

If the roots are real & equal.

$a_n = (A + Bn)k^n$

case (iii):-

If the roots are real & imaginary.

i.e., $p+q$

$a_n = r^n [A\cos\theta + B\sin\theta]$

where $\theta = \tan^{-1}\left[\frac{q}{p}\right]$

$r = \sqrt{p^2+q^2}$.

1) Solve the R.R $a_n + a_{n-1} + 6a_{n-2} = 0$ for $n \geq 2$ and $a_0 = -1$, $a_1 = 8$.

Given

$a_n + a_{n-1} - 6a_{n-2} = 0$ —— ①

By 2nd order linear (or) H.R.R.

$c^n a_n + c^{n-1} a_{n-1} + c^{n-2} a_{n-2} = 0$ for $n \geq 2$ —— ②.

Comparing all the co-efficient in eq ① & ②.

$c^n = 1$, $c^{n-1} = 1$, $c^{n-2} = 6$.

sub $a_n = k^n$ in eqn ②

$c^n k^n + c^{n-1} k^{n-1} + c^{n-2} k^{n-2} = 0$

$(1)k^n + (1)k^{n-1} + (-6)k^{n-2} = 0$.

$k^n + k^{n-1} + (-6)k^{n-2} = 0$.

$k^n + k^{n-1} - 6k^{n-2} = 0$.

For $n=2$

$$k^2+k^{2-1}-6k^{2-2}=0.$$

$$k^2+k-6=0$$

$$k^2-2k+3k-6=0.$$

$$k(k-2)+3(k-2)=0 \quad \Rightarrow (k-2)(k+3)=0.$$

$$\Rightarrow (k-2)=0 \quad (k+3)=0.$$

$$\Rightarrow k=2, -3.$$

Let $k_1=2$, $k_2=-3$.

The roots are real & distinct.

$$a_n = A(k_1)^n + B(k_2)^n$$

$$a_n = A(2)^n + B(-3)^n \quad \text{— ③}.$$

If $n=0$.

$$a_0 = A(2)^0 + B(-3)^0$$

$$-1 = A+B.$$

$$A+B=-1 \quad \text{— ④}.$$

If $n=1$

$$a_1 = A(2)^1 + B(-3)^1$$

$$B = 2A-3B$$

$$2A-3B = B \quad \text{— ⑤}$$

Solve eq ④ × ③ by ⑤.

$$3A+\cancel{3B}=-3.$$
$$2A-\cancel{3B}=8$$
$$5A=5$$
$$\boxed{A=1}$$

From ④.

$$A+B=-1$$
$$1+B=-1$$
$$\boxed{B=-2}$$

Sub A, B values in eqn ③.

$$a_n = (1)(2)^n + (-2)(-3)^n.$$

2) Solve. R.R $a_n = 2[a_{n-1} - a_{n-2}]$ for $n\geq 2$ and $a_0=1$, $a_1=2$

Given:-

$$a_n = 2[a_{n-1} - a_{n-2}] \quad \text{— ①}.$$

$$a_n - 2a_{n-1} + 2a_{n-2} = 0.$$

By 2^{nd} order (or) H.R.R is

$$c^n a_n + c^{n-1} a_{n-1} + c^{n-2} a_{n-2} = 0 \text{ for } n\geq 2 \quad \text{— ②}$$

comparing all co-eff in eqn ① & ②

$$c^n = 1, \quad c^{n-1}=2, \quad c^{n-2}=2$$

Sub $a_n = k^n$ in eq (2).

$c^n k_n + c^{n-1} k_{n-1} + c^{n-2} k_{n-2} = 0.$

$(1) k_n + (-2) k^{n-1} + (2) k^{n-2} = 0.$

$k^n - 2k^{n-1} + 2k^{n-2} = 0.$

for $n = 2$.

$k^2 - 2k^{2-1} + 2k^{2-2} = 0.$

$k^2 - 2k + 2k^0 = 0.$

$k^2 - 2k + 2 = 0.$

Here $a=1,\ b=-2,\ c=2$

$$k = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} \Rightarrow \frac{2 \pm \sqrt{4-8}}{2} \Rightarrow \frac{2 \pm \sqrt{4(-1)}}{2}$$

$$\Rightarrow \frac{2 \pm \sqrt{4}\sqrt{-1}}{2} \Rightarrow \frac{2 \pm 2i}{2} \Rightarrow \frac{\cancel{2}(1 \pm i)}{\cancel{2}}$$

$\left[\sqrt{ab} = \sqrt{a}\cdot\sqrt{b},\ i = \sqrt{-1},\ i^2 = -1 \right]$

$k \Rightarrow 1 \pm i.$

The roots are real and imaginary.

$a_n = r^n [A \cos\theta + B \sin\theta]$ — (3)

$p + iq = 1 \pm i.$

Here $p=1$ $q=1$

$r = \sqrt{p^2+q^2} = \sqrt{1^2+1^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{q}{p}\right) \Rightarrow \tan^{-1}\left(\frac{1}{1}\right).$

$= \tan^{-1}(1) \Rightarrow \tan^{-1}\left(\tan\frac{\pi}{4}\right)$

$\theta = \frac{\pi}{4}$

From (3)

$a_n = (\sqrt{2})^n \left[A \cos\frac{\pi}{4} + B \sin\frac{\pi}{4}\right]$

If $n=0$

$a_0 = (\sqrt{2})^0 \left[A \cos(0)\frac{\pi}{4} + B \sin(0)\frac{\pi}{4}\right]$

$a_0 = (1) [A \cos(1) + B \sin(1)]$

$1 = A(1) + B(1)$

$A = 1$

If $n=1$

$a_1 = (\sqrt{2}) \left[A \cos(1)\frac{\pi}{4} + B \sin(1)\frac{\pi}{4}\right]$

$2 = \sqrt{2}\left[A\cos\frac{\pi}{4} + B\sin\frac{\pi}{4}\right]$

$= \sqrt{2}\left[A\frac{1}{\sqrt{2}} + B\frac{1}{\sqrt{2}}\right]$

$= \sqrt{2}\left[\frac{A+B}{\sqrt{2}}\right]$

$2 = A+B$

$A+B = 2$

$1+B = 2$

$\boxed{B = 1}$

from ③

$a_n = (\sqrt{2})^n\left[(1)\cos\frac{n\pi}{4} + (1)\sin\frac{n\pi}{4}\right]$

3) Solve the R.R $a_n - 6a_{n-1} + 9a_{n-2} = 0$ for $n \geq 2$, $a_0 = 5$, $a_1 = 12$

Given :-

$a_n - 6a_{n-1} + 9a_{n-2} = 0$ —①.

By 2^{nd} order (or) H.R.R. is

$c^n a_n + c^{n-1} a_{n-1} + c^{n-2} a_{n-2} = 0$ for $n \geq 2$ —②

Comparing all co.eff of eqn ① & ②.

$c^n = 1$, $c^{n-1} = -6$, $c^{n-2} = 9$

Sub $a_n = k^n$ in eq②

$c^n k^n + c^{n-1} k^{n-1} + c^{n-2} k^{n-2} = 0$.

$(1)k^n - 6k^{n-1} + 9k^{n-2} = 0$.

$k^n - 6k^{n-1} + 9k^{n-2} = 0$.

for $n = 2$

$k^2 - 6k^{2-1} + 9k^{2-2} = 0$.

$k^2 - 6k + 9k^0 = 0$.

$k^2 - 6k + 9 = 0$.

$k^2 - 3k - 3k + 9 = 0$.

$k(k-3) - 3(k-3) = 0$

$k = 3, 3$.

roots are real & equal.

$a_n = (A+Bn)k^n$

$= (A+Bn)3^n$ —③

If $n=0$.

$a_0 = (A+B(0))3^0$

$a_0 = (A+0)3^0$

$a_0 = A$

$\boxed{A=5}$

If $n=1$

$a_1 = (A+B(1))3^1$

$= (A+B)3$

$12 = (5+B)3$

$12 = 15+3B \Rightarrow -3 = 3B$

$B = -1$

~~from~~ eq③

$a_n = (5-n)3^n$

4) Find the R.R $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$, $F_0 = 0$, $F_1 = 1$.

Given :-

$F_{n+2} = F_{n+1} + F_n$

put $n = n-2$ in the order eqn.

$F_{n-2+2} = F_{n-2+1} + F_{n-2}$

$F_n = F_{n-1} + F_{n-2}$

$F_n - F_{n-1} - F_{n-2} = 0$ — ①

By 2nd order liner (or) H R.R

$c^n F_n + c^{n-1} F_{n-1} + c^{n-2} F_{n-2} = 0$ — ②

comparing all co-ef in eqn ① & ②

$c^n = 1$, $c^{n-1} = -1$, $c^{n-2} = -1$.

sub $F_n = k^n$ in eq ②

$c^n k^n + c^{n-1} k^{n-1} + c^{n-2} k^{n-2} = 0$.

$(1)k^n + (-1)k^{n-1} + (-1)k^{n-2} = 0$.

$k^n - k^{n-1} - k^{n-2} = 0$.

for $n=2$

$$k^2 - k^{2-1} - k^{2-2} = 0.$$

$$k^2 - k - 1 = 0.$$

Here $a=1$, $b=-1$, $c=-1$.

$$k = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} \Rightarrow \frac{1 \pm \sqrt{1+4}}{2} \Rightarrow \frac{1 \pm \sqrt{5}}{2}$$

$$k_1 = \frac{1+\sqrt{5}}{2}, \quad k_2 = \frac{1-\sqrt{5}}{2}$$

The roots are real and distinct.

$$f_n = A(k_1)^n + B(k_2)^n$$

$$f_n = A\left[\frac{1+\sqrt{5}}{2}\right]^n + B\left[\frac{1-\sqrt{5}}{2}\right]^n. \quad \text{--- (3)}$$

If $n=0$.

$$F_0 = A\left[\frac{1+\sqrt{5}}{2}\right]^0 + B\left[\frac{1-\sqrt{5}}{2}\right]^0$$

$$0 = A(1) + B(1).$$

$$A + B = 0 \quad \Rightarrow A = -B \quad \text{--- (4)}.$$

If $n=1$

$$F_1 = A\left[\frac{1+\sqrt{5}}{2}\right]^1 + B\left[\frac{1-\sqrt{5}}{2}\right]^1$$

$$1 = \frac{A(1+\sqrt{5}) + B(1-\sqrt{5})}{2}$$

$$1 = \frac{-B(1+\sqrt{5}) + B(1-\sqrt{5})}{2} \Rightarrow 2 = \cancel{-B} - \sqrt{5}B \cancel{+B} - \sqrt{5}B$$

$$\Rightarrow 2 = -2\sqrt{5}B \Rightarrow B = \frac{\cancel{2}}{-\cancel{2}\sqrt{5}} \Rightarrow B = -\frac{1}{\sqrt{5}}$$

from (4).

$$A = -B$$

$$= -\left[-\frac{1}{\sqrt{5}}\right]$$

$$A = \frac{1}{\sqrt{5}}$$

from (3)

$$f_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n + \left(-\frac{1}{\sqrt{5}}\right)\left[\frac{1-\sqrt{5}}{2}\right]^n.$$

Third order linear (or) homogeneous R.R:-

The third order linear (or) HRR with constant co-ef as the following form.

$$c^n a_n + c^{n-1} a_{n-1} + c^{n-2} a_{n-2} + c^{n-3} a_{n-3} = 0 \quad \text{for } n \geq 3.$$

Proof :-

Given

$$c^n a_n + c^{n-1} a_{n-1} + c^{n-2} a_{n-2} + c^{n-3} a_{n-3} = 0 \quad \text{for } n \geq 3 \quad \text{--- (1)}$$

Sub $a_n = k^n$ in eqn (1)

$$c^n k^n + c^{n-1} k^{n-1} + c^{n-2} k^{n-2} + c^{n-3} k^{n-3} = 0.$$

for $n=3$.

$$c^3 k^3 + c^{3-1} k^{3-1} + c^{3-2} k^{3-2} + c^{3-3} k^{3-3} = 0.$$

$$c^3 k^3 + c^2 k^2 + ck + ck^0 = 0.$$

$$c^3 k^3 + c^2 k^2 + ck + 1 = 0.$$

Case (i) :-

If the roots are real and distinct.

$$a_n = A(k_1)^n + B(k_2)^n + C(k_3)^n$$

Case (ii) :-

If the roots are real and equal.

$$a_n = (A + Bn + Cn^2) k^n.$$

case (iii) :-

If the roots are real and imaginary

i.e., $p + iq$

$$a_n = A(k_1)^n + r^n [A\cos\theta + B\sin\theta]$$

where $r = \sqrt{p^2+q^2}$

$$\theta = \tan^{-1}\left(\frac{q}{p}\right).$$

1) Solve the R.R $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ for $n \geq 3$ & $a_0 = 1$, $a_1 = 4$, $a_2 = 28$.

Given :-

$$a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0 \quad \text{--- (1)}$$

By 3rd order linear (or) H.R.R

$$c^n a_n + c^{n-1} a_{n-1} + c^{n-2} a_{n-2} + c^{n-3} a_{n-3} = 0 \quad \text{--- (2)}$$

Comparing all the co-ef in eqn (1) & (2).

$c^n = 1$, $c^{n-1} = -6$, $c^{n-2} = 12$, $c^{n-3} = -8$.

Sub $a_n = k^n$ in eqn ②

$c^n k^n + c^{n-1} k^{n-1} + c^{n-2} k^{n-2} + c^{n-3} k^{n-3} = 0$.

$(1) k^n + (-6) k^{n-1} + 12 k^{n-2} + (-8) k^{n-3} = 0$

$k^n - 6k^{n-1} + 12 k^{n-2} - 8k^{n-3} = 0$.

For $n = 3$.

$k^3 - 6k^{3-1} + 12k^{3-2} - 8k^{3-3} = 0$

$k^3 - 6k^2 + 12k - 8 = 0$.

$$\begin{array}{c|cccc} k=2 & 1 & -6 & 12 & -8 \\ & 0 & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$k^2 - 4k + 4 = 0$.

$k^2 - 2k - 2k + 4 = 0$.

$k(k-2) - 2(k-2) = 0$.

$(k-2)(k-2) = 0$.

$k = 2, 2$

$k = 2, 2, 2$

The roots are real & equal.

$a_n = [A + Bn + Cn^2] k^n$

$a_n = [A + Bn + Cn^2] (2)^n$ —— ③

If $n = 0$

$a_0 = (A + B(0) + C(0)^2)(2)^0$

$\boxed{A = 1}$

If $n = 1$

$a_1 = (A + B(1) + C(1)^2)(2)^1$

$4 = (A+B+C) 2$

$1 + B + C = 2$

$B + C = 1$ —— ④

If $n = 2$

$a_2 = [A + B(2) + C(2)^2](2)^2$

$28 = [A + 2B + 4C] 4$

$7 = 1 + 2B + 4C \Rightarrow 2B + 4C = 6. \Rightarrow 2(B + 2C) = 6 \Rightarrow B + 2C = 3$ —— ⑤

Solve eqn ④ & ⑤

$$B + C = 1$$
$$-\underline{B + 2C = 3}$$
$$-C = -2$$

$$\boxed{C = 2}$$

from ④

$$B + C = 1$$
$$B + 2 = 1$$
$$B = -1$$

from ③

$$a_n = [1 + (-1)n + 2n^2](2)^n.$$

2) Solve the R.R $2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n$ for $n \geq 0$, $a_0 = 0$, $a_1 = 1$, $a_2 = 2$.

Given:-

$$2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n. \quad —① \quad \text{for } n \geq 0$$

$$a_0 = 0, \ a_1 = 1, \ a_2 = 2$$

Sub n by n-3 in eqn ①

$$2a_{n-3+3} = a_{n-3+2} + 2a_{n-3+1} - a_{n-3}.$$

$$2a_n = a_{n-1} + 2a_{n-2} - a_{n-3}$$

$$2a_n - a_{n-1} - 2a_{n-2} + a_{n-3} = 0. \quad —②$$

By 3rd order linear or H. R. R is

$$c^n a_n + c^{n-1} a_{n-1} + c^{n-2} a_{n-2} + c^{n-3} a_{n-3} = 0 \quad —③$$

Comparing all the co-efficients

$$c^n = 2, \ c^{n-1} = -1, \ c^{n-2} = -2, \ c^{n-3} = 1$$

Sub $a_n = k^n$ in eqn ③

$$c^n k^n + c^{n-1} k^{n-1} + c^{n-2} k^{n-2} + c^{n-3} k^{n-3} = 0.$$

$$(2)k^n + (-1)k^{n-1} + (-2)k^{n-2} + (1)k^{n-3} = 0.$$

$$2k^n - k^{n-1} - 2k^{n-2} + k^{n-3} = 0.$$

for n = 3

$$2k^3 - k^{3-1} - 2k^{3-2} + k^{3-3} = 0.$$

$$2k^3 - k^2 - 2k + 1 = 0.$$

$$\begin{array}{r|rrrr} k=1 & 2 & -1 & -2 & 1 \\ & 0 & 2 & 1 & -1 \\ \hline & 2 & 1 & -1 & \boxed{0} \end{array}$$

$2k^2+k-1=0.$

$2k^2+2k-k-1=0.$

~~$2k[k+1]$~~ $2k[k+1]-1[k+1]=0.$

$k+1=0, \quad 2k-1=0.$

$k=-1, \quad 2k=1 \Rightarrow k=1/2.$

$\therefore k=+1, -1, 1/2.$

The roots are real & distinct

$a_n = A(k_1)^n + B(k_2)^n + C(k_3)^n$

$a_n = A(1)^n + B(-1)^n + C(1/2)^n$ —— ④

If $n=0$.

$a_0 = A(1)^0 + B(-1)^0 + C(1/2)^0$

$0 = A+B+C$ —— ⑤

If $n=1$

$a_1 = A(1)^1 + B(-1)^1 + C(1/2)^1.$

$1 = A-B+1/2 \Rightarrow 1 = \frac{2A-2B+C}{2} \Rightarrow 2A-2B+C=2$ —— ⑤

If $n=2$

$a_2 = A(1)^2 + B(-1)^2 + C(1/2)^2$

$2 = A+B+C(1/4) \Rightarrow 2 = \frac{4A+4B+C}{4} \Rightarrow 4A+4B+C=8$ —— ⑥

Solve eqn ⑤×4 and ⑦.

$$\begin{array}{r} \cancel{4A}+\cancel{4B}+4C=0. \\ \cancel{4A}+\cancel{4B}-C=8. \\ - \quad - \quad + \\ \hline 3C=-8. \\ C=-8/3 \end{array}$$

Solve eqn ⑥×2 & ③

$$\begin{array}{r} \cancel{4A}-\cancel{4B}+2C=4 \\ \cancel{4A}+\cancel{4B}+C=8 \\ \hline 8A+3C=12. \end{array}$$

$8A+\cancel{3}(-\frac{8}{\cancel{3}})=12.$

$8A-8=12$

$8A=20.$

$A=\frac{20}{8}$

$A=\frac{5}{2}$

from ⑤.

$A+B+C=0.$

$\frac{5}{2}+B+(-\frac{8}{3})=0 \Rightarrow \frac{15+6B-16}{6}=0 \Rightarrow 6B-1=0 \Rightarrow B=1/6$

from ④.

$$a_n = A(1)^n + B(-1)^n + C(1/2)^n$$

$$a_n = \frac{5}{2}(1)^n + \frac{1}{6}(-1)^n - \frac{8}{3}(1/2)^n.$$

3) Solve the R.R $a_n + a_{n-3} = 0$ for $n \geq 3$

Given:-

$$a_n + (0)a_{n-1} + (0)a_{n-2} + a_{n-3} = 0 \quad \text{—— ①}$$

By 3^{rd} order linear (or) Homogeneous R.R is

$$c^n a_n + c^{n-1} a_{n-1} + c^{n-2} a_{n-2} + c^{n-3} a_{n-3} = 0 \quad \text{—— ②}$$

Comparing all the co-eff in eqn ① & ②.

$c^n = 1, \quad c^{n-1} = 0, \quad c^{n-2} = 0, \quad c^{n-3} = 1$

Sub $a_n = k^n$ in eqn ②

$$c^n k^n + c^{n-1} k^{n-1} + c^{n-2} k^{n-2} + c^{n-3} k^{n-3} = 0.$$

$$(1)k^n + (0)k^{n-1} + (0)k^{n-2} + (0)k^{n-3} = 0.$$

$$k^n + k^{n-3} = 0.$$

for $n = 3$

$$k^3 + k^{3-3} = 0$$

$$k^3 + k^0 = 0.$$

$$k^3 + (1) = 0.$$

$$k^3 + 1^3 = 0.$$

$$\{a^3 + b^3 = (a+b)(a^2 - ab + b^2)\}$$

$$(k+1)(k^2 - k(1) + (1)^2) = 0.$$

$k+1 = 0, \quad k^2 - k + 1 = 0.$

$k = -1$, Here $a = 1, b = -1, c = 1 \Rightarrow k = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$k = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1-4}}{2} \Rightarrow \frac{1 \pm \sqrt{-3}}{2} \Rightarrow \frac{1 \pm \sqrt{3(-1)}}{2}$$

$$\Rightarrow \frac{1 \pm \sqrt{3}\sqrt{-1}}{2} \Rightarrow \frac{1 \pm \sqrt{3}i}{2} \quad \& \; k = -1$$

The roots are real and imaginary

$$a_n = A(k_1)^n + r^n[A\cos n\theta + B\sin n\theta]$$

$$a_n = A(-1)^n + r^n[A\cos n\theta + B\sin n\theta] \quad \text{—— ③.}$$

$\therefore p+iq = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

Here $p = 1/2$, $q = \frac{\sqrt{3}}{2}$

$r = \sqrt{p^2+q^2} = \sqrt{(1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \Rightarrow \sqrt{\frac{1}{4}+\frac{3}{4}} \Rightarrow \sqrt{\frac{4}{4}}$

$r = 1$

$\theta = \tan^{-1}\left(\frac{q}{p}\right)$

$= \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) \Rightarrow \tan^{-1}(\sqrt{3})$

$\Rightarrow \tan^{-1}\left(\tan\frac{\pi}{3}\right)$

$\theta = \frac{\pi}{3}$

~~from~~ ③:

$a_n = A(-1)^n + (1)^n \left[A\cos n\frac{\pi}{3} + B\cos n\frac{\pi}{3}\right]$

Inclusion, Exclusion Principle :-

Let a, b are any two finite sets

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$. Here $n(A)$, $n(B)$ are inclusion and $n(A \cap B)$ is exclusion.

NOTE :-

* Suppose A, B, C are any three finite sets then

$n(A\cup B\cup C) = n(A)+n(B)+n(C) - n(A\cap B) - n(A\cap C) - n(B\cap C) + n(A\cap B\cap C)$

* A formula for the no. of elements in the union of 4 sets are A_1, A_2, A_3, A_4 then

$n(A_1\cup A_2\cup A_3\cup A_4) = n(A_1)+n(A_2)+n(A_3)+n(A_4) - n(A_1\cap A_2) - n(A_1\cap A_3)$
$- n(A_1\cap A_4) - n(A_2\cap A_3) - n(A_2\cap A_4) - n(A_3\cap A_4) + n(A_1\cap A_2\cap A_3\cap A_4)$

1) Given in a grp of 42 tourist everyone speaks english or french; there are 35 English speakers and 18 French Speakers. How many speak both the English and french.

$n(A\cup B) = 42$
$n(A) = 35$
$n(B) = 18$
$n(A\cap B) = ?$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$= 35 + 18 - 42$

$= 53 - 42$

$= 11.$

2) Consider a set of integers from 1 to 250 find out

i) How many of this no. are divisible by 3 or 5.

ii) How many of this no. are divisible by 5 or 7

iii) How many of this no. are divisible by 3 or 7.

iv) How many of this no. are divisible by 3 or 5 or 7

v) How many of this no. are divisible by 3 or 5 but not by 7.

vi) How many of this no. are divisible by 5 or 7 but not by 3.

vii) How many of this no. are divisible by 3 or 7 but not by 5.

viii) How many of these integers b/w 1 and 250 not divisible by 3, 5, 7.

Let A_1, A_2, A_3 are the sets consisting of integers that are divisible by 3, 5, 7 resp.

$n(A_1) = \frac{250}{3} = 83$

$n(A_2) = \frac{250}{5} = 50$

$n(A_3) = \frac{250}{7} = 35$

$n(A_1 \cap A_2) = \frac{250}{3 \times 5} = 16$

$n(A_1 \cap A_3) = \frac{250}{3 \times 7} = 11$

$n(A_2 \cap A_3) = \frac{250}{5 \times 7} = 7$

$n(A_1 \cap A_2 \cap A_3) = \frac{250}{3 \times 5 \times 7} = 2.$

i) $n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2)$

$= 83 + 50 - 16.$

$= 83 + 34$

$= 117$

ii) $n(A_2 \cup A_3) = n(A_2) + n(A_3) - n(A_2 \cap A_3)$

$= 50 + 35 - 7$

$= 50 + 28 \quad = 78$

iii) $n(A_1 \cup A_3) = n(A_1) + n(A_3) - n(A_1 \cap A_3)$

$= 83 + 35 - 11$

$= 107$

iv) $n(A_1 \cup A_2 \cup A_3) = n(A_1) + n(A_2) + n(A_3) - n(A_1 \cap A_2) - n(A_1 \cap A_3) - n(A_2 \cap A_3) + n(A_1 \cap A_2 \cap A_3)$.

$= 83 + 50 + 35 - 16 - 11 - 7 + 2$

$= 136.$

v) $n(A_1 \cup A_2 \cup A_3) - n(A_3)$

$= 136 - 35$

$= 101.$

vi) $n(A_1 \cup A_2 \cup A_3) - n(A_1)$

$= 136 - 83$

$= 53.$

vii) $n(A_1 \cup A_2 \cup A_3) - n(A_2)$

$= 136 - 50$

$= 86.$

viii) $250 - n(A_1 \cup A_2 \cup A_3)$.

$= 250 - 136$

$= 114.$

3) A servey an sample on 25 new cars being sold out at a local auto dealer was conducted to see which of 3 popular options air conditioner (A), radio(R), power windows (W) where already installed. The survey found 15 had air conditioners, 12 had radios, 11 had power windows, 5 had air conditioner and power windows, 9 had air conditioner & radios, 4 had radio and power windows. 3 had all three options. find no. of cars which had i) only one of the option.

ii) atleast one of the option.

iii) None of these options. use principle of inclusion and exclusion

Given :- $n(A) = 15$

$n(R) = 12$

$n(W) = 11$

$n(A \cap R) = 9$

$n(A \cap W) = 5$

$n(R \cap W) = 4$

$n(A \cap R \cap W) = 3.$

$n(A \cup R \cup W) = n(A) + n(R) + n(W) - n(A \cap R) - n(A \cap W) - n(R \cap W)$

$= 15 + 12 + 11 - 9 - 5 - 4 + 3$

$= 41 - 18$

$= 23$

No. of cars that had none of the options $= 25 - 23 = 2$

Divide and conquer algorithm:-

It works recursively breaking down a problem into two or more subproblems of the sum are related type until this becomes simple enough to be solved directly the soln to the subproblem (or) combined to give a soln to the original prblm.

A typical divide & conquer algorithm solve a problem using the three steps :—

i) Divide:-

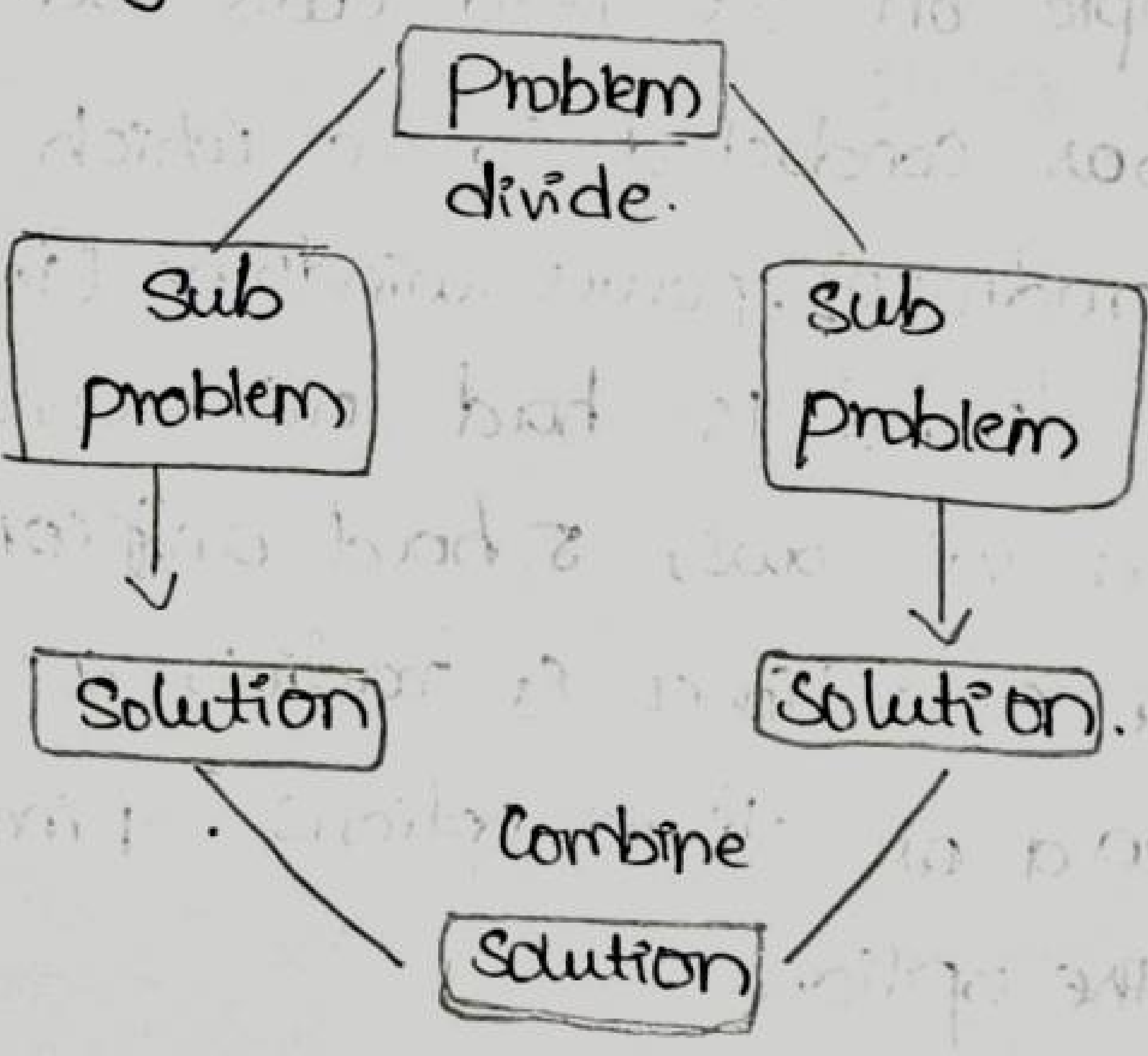
Break the given prblm into two or more sub prblms of same ~~type~~ time.

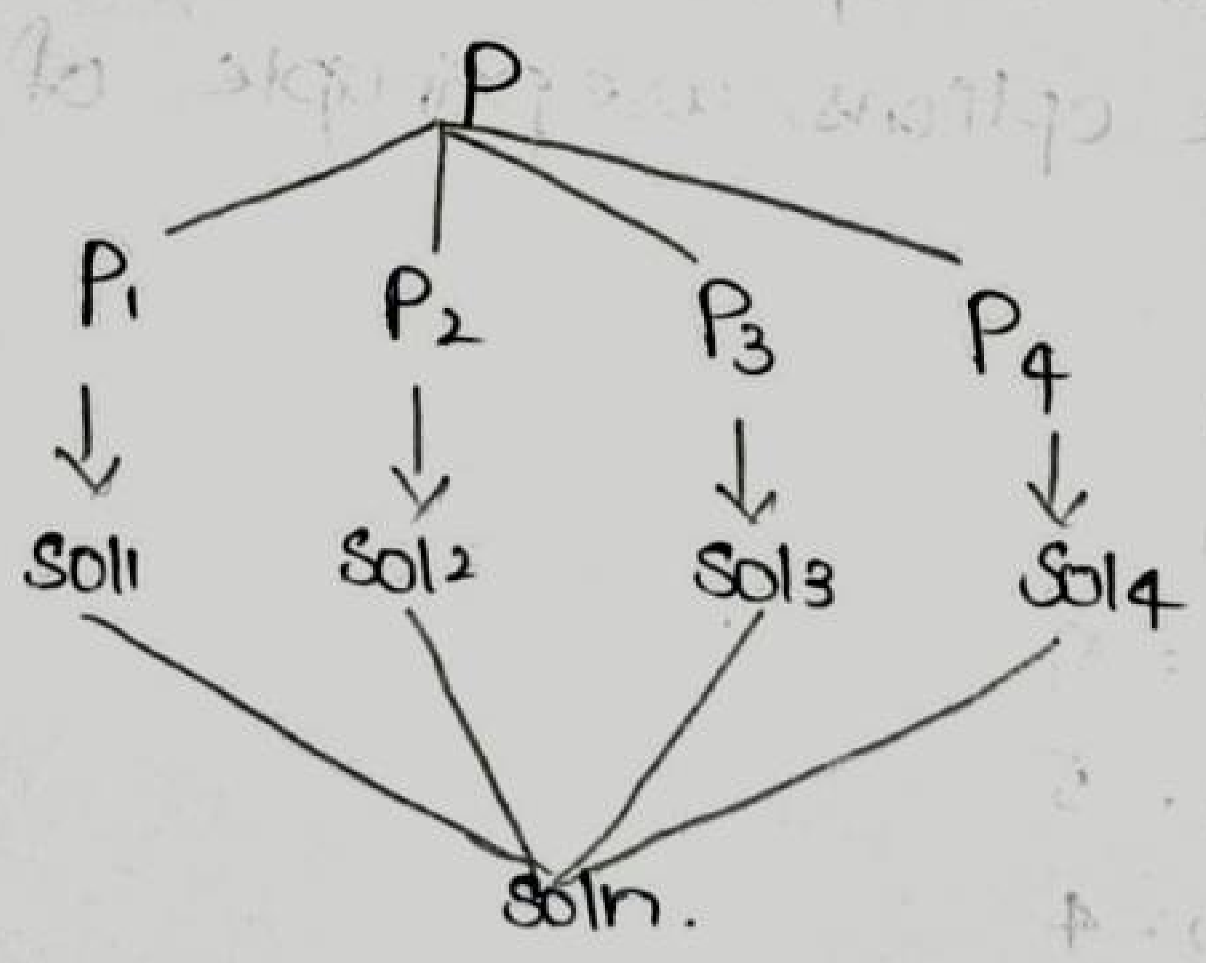
ii) Conquer :-

Recursively solve the sub-prblms.

iii) Combine :-

Approximately combine the ans.





UNIT - 5

Graphs

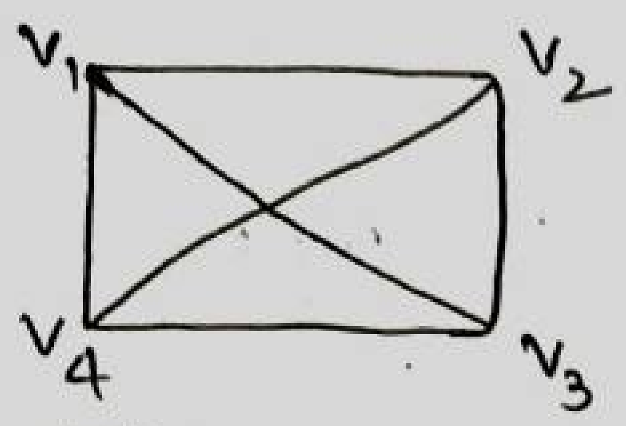
Simple (or) Complete Graph :-

A simple graph 'G' is said to be a complete graph, if there is a single edge b/w each and every pair of variables.

→ Graph doesn't Combines any self loop.

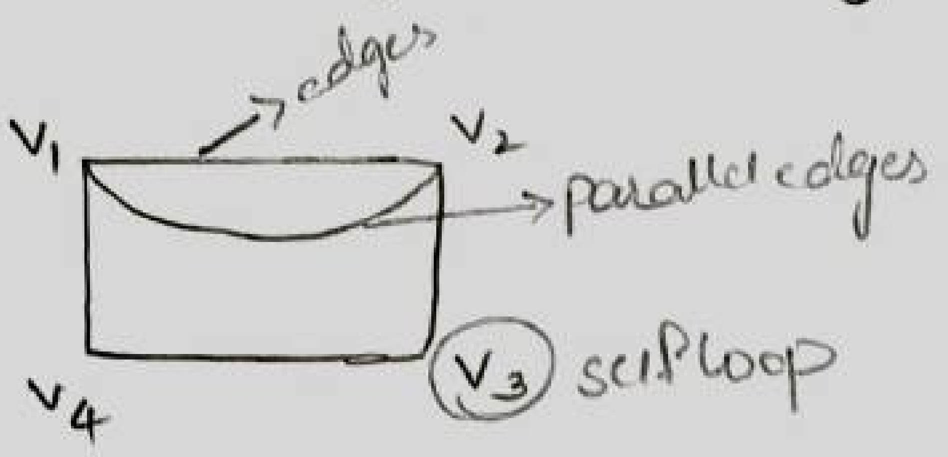
→ No parallel edges.

→ In complete graph $\frac{n(n-1)}{2}$ No. of edges.



edges

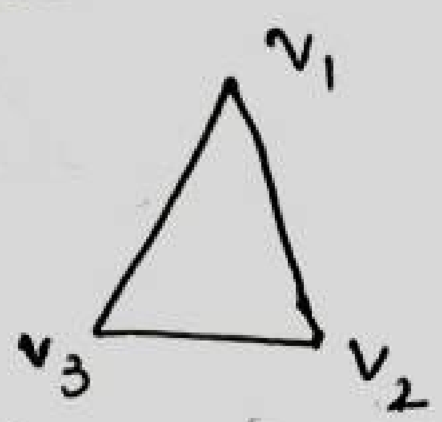
$v_1 = 3$ $v_2 = 3$ $v_3 = 3$ $v_4 = 3$.



Regular graph :-

→ All the vertices of equal degree is called regular graph.

→ It is of degree n-1.



$v_1 = 2$
$v_2 = 2$
$v_3 = 2$

∴ The degree is $(n-1) = 3-1 = 2$

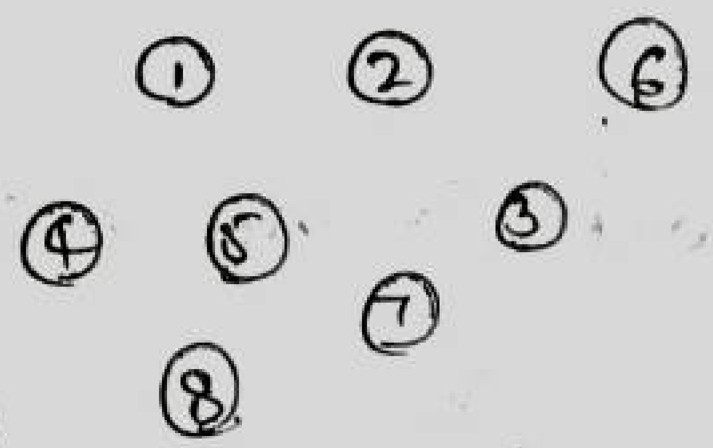
Null Graph:-

→ It is a graph in which there are no edges b/w the vertices of the graph.

→ It is a empty graph.

→ It contains only isolated nodes

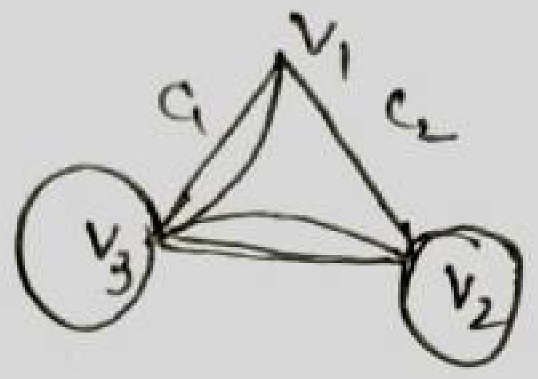
→ set of edges in null graph is zero.



edges = No
only vertices

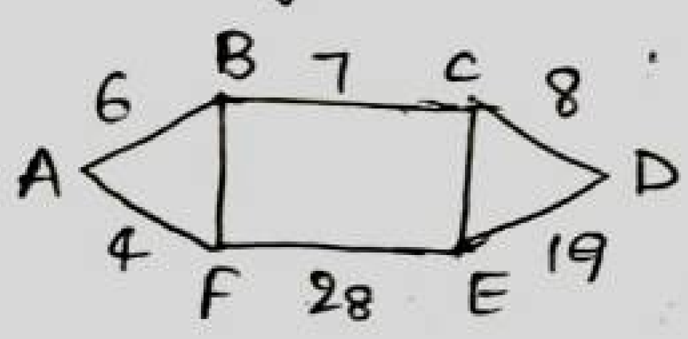
Multi Graph:-

A Graph which contain some parallel edges & self loops is call multi Graph.



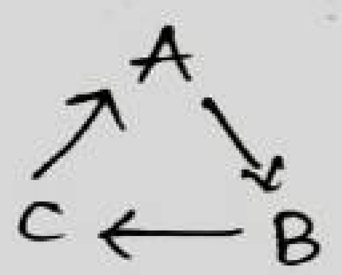
Waited Graph:-

A Graph in which cost (or) some value are signed to every edge of the graph is called waited graph.



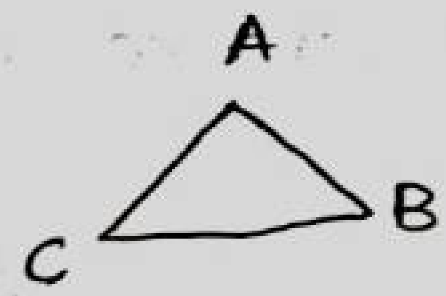
Directed Graph:-

Every edge is direct is called direct Graph.



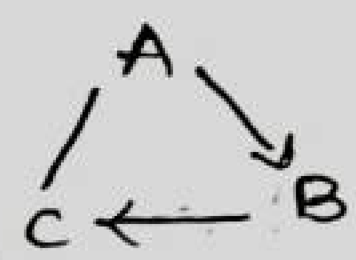
Undirected Graph:-

A Graph in every edge is not directed is called undirected Graph.



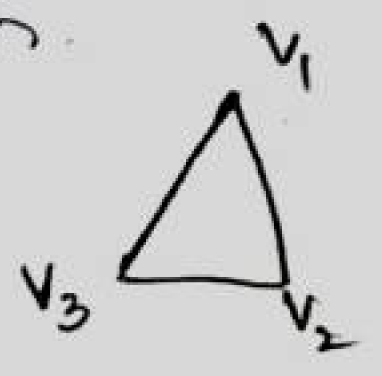
Mixed Graph:-

A Graph in which some edges are directed and some edges are undirected is called mixed graph.

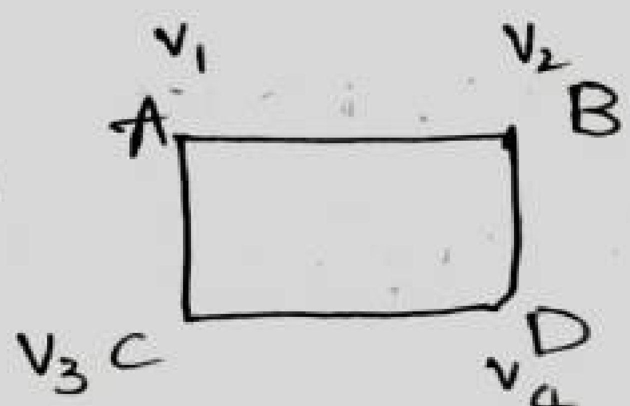


Cycle Graph:-

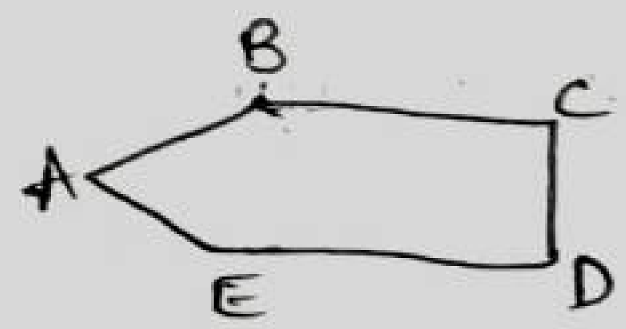
A Graph Contain atleast one Cycle in it is called as Cycle graph.



(or)

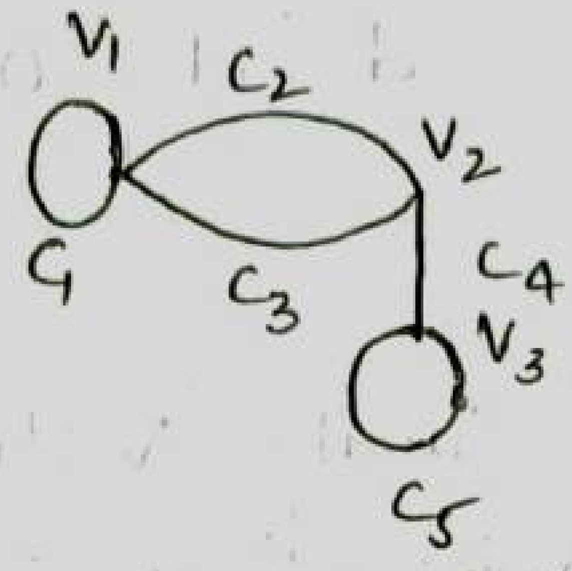


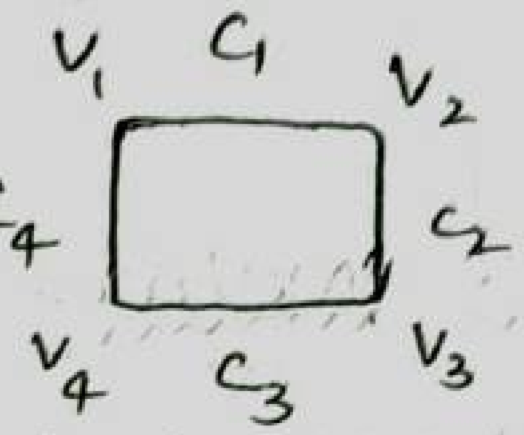
(or)



$v_1 = 2$
$v_2 = 2$
$v_3 = 2$

$v_1 = 2$
$v_2 = 2$
$v_3 = 2$
$v_4 = 2$





	c_1	c_2	c_3	c_4	c_5
v_1	1	1	1	0	0
v_2	0	1	1	1	0
v_3	0	0	0	1	1

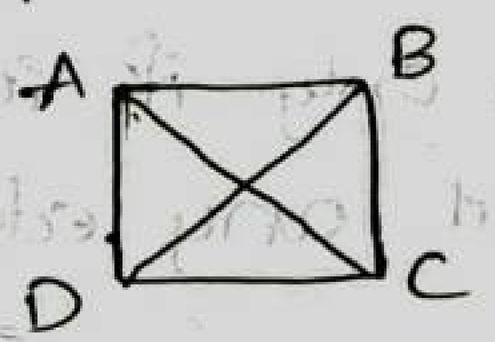
	c_1	c_2	c_3	c_4
v_1	1	0	0	1
v_2	1	1	0	0
v_3	0	1	1	0
v_4	0	0	1	1

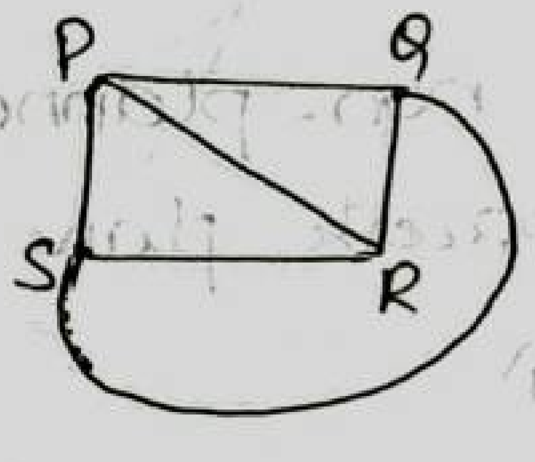
Isomorphism of Graph theory:-

Consider 2 Graphs $G(V, E)$ & $G'(V', E')$. Suppose their exist a function.

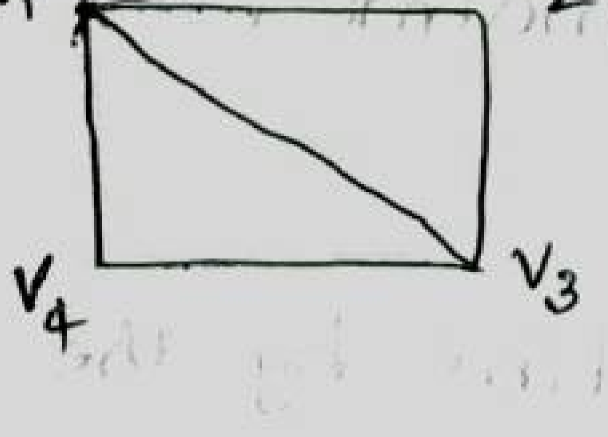
1) $F: V \to V'$ ~~search~~ such that F is one-one
2) the edge $\{A, B\} \in E$, if-&-only-if the edge $\{F(A), F(B)\} \in E$

then f is called an isomorphism b/w G & G' (G, G') are isomorphic graphs.



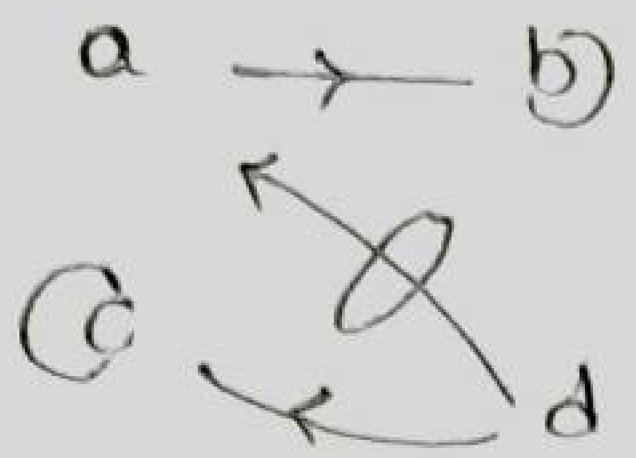


A-B = P-Q
B-C = Q-R
C-D = R-S
D-A = S-P
A-C = P-R
B-D = Q-S



v_i v_j

	v_1	v_2	v_3	v_4
v_1	0	1	1	1
v_2	1	0	1	0
v_3	1	1	0	1
v_4	1	0	1	0



$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 0 \\ b & 0 & 1 & 1 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 1 & 0 & 1 & 0 \end{array}$$

Incidence Matrix:-

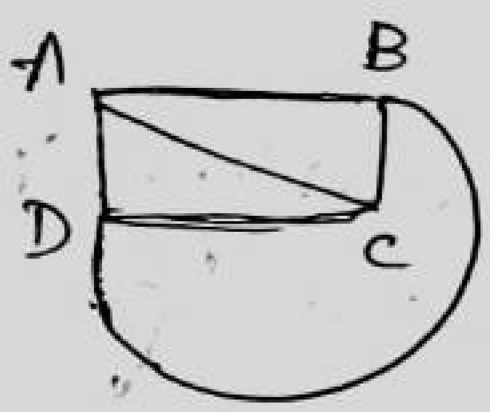
let $G(V,E)$ is undirected graph with vertices order from $v_1, v_2, v_3 \dots v_n$ and 'm' edges order from $e_1, e_2 \dots e_m$ then the incidence of representation of order is $n \times m$

$$I_n = [m_{ij}]_{m \times n}$$

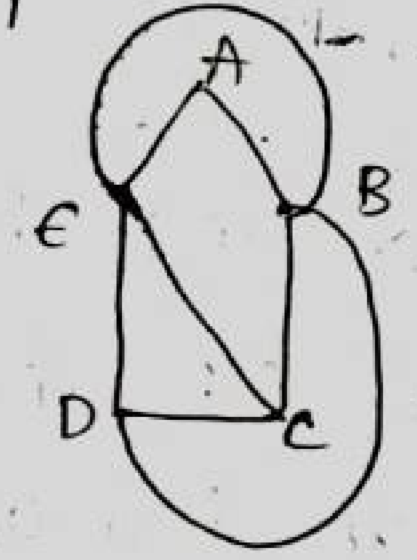
$m_{ij} = \{1$, when edge e_j is incident on v_i 0 other wise$\}$.

Plannar Graph:-

A graph is said to be plannar, if & only if it can be drawn in a separate plane without any edge cross.

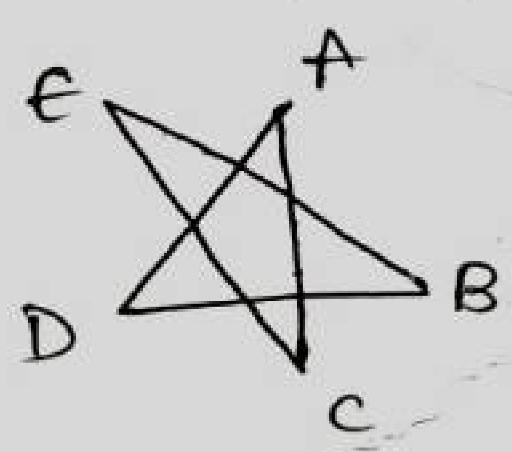


(or)



Non-plannar graph:-

G is said to be non-plannar if & only if it is connot be drawn in separate plane without any edge crossing.



Matrix Representation of Graph:-

Adjacency:-

→ Let $G(V,E)$ is the simple graph with 'n' vertices order from v_1 to v_n then the adjacency matrix is :

$$A = [a_{ij}]_{m \times n}.$$

→ G is an $m \times n$ symmetric matrix define by the following columns

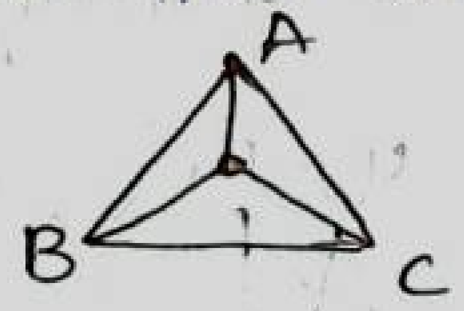
a_{ij} = {1 when v_i is adjacences to v_j 0 otherwise}

(or)

a_{ij} = { If 1 v_i & v_j are connected by edge 0 otherwise}

Wheel Graph :-

→ It is a cycle (n) & cycle graph formed with (n-1) vertices and an extra single vertix edges adjaced end to every vertix into in the edge (wn)

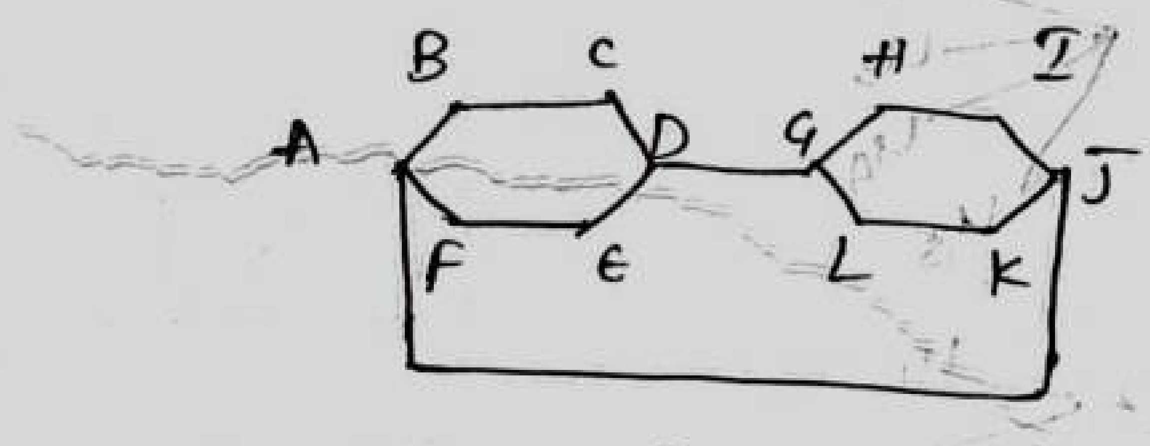


Star Graph :-

It is a spe type of graph in which (n-1) vertices have degree one and a single vertices.

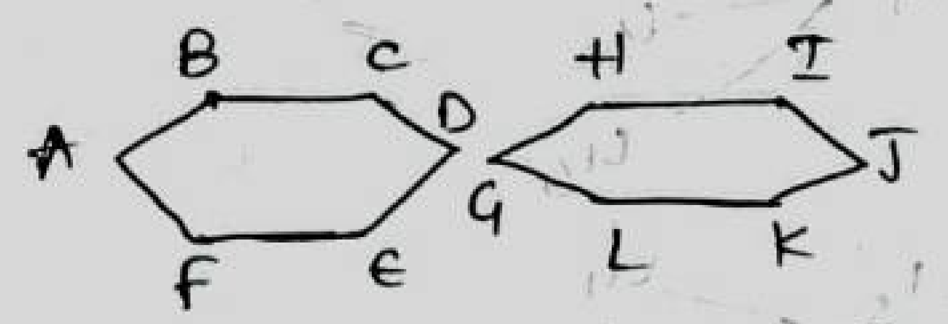
Connected Graph :-

A graph is said to be connected graph, if ∃ a path (or) atleast one edge b/w every pair of vertices in the given graph.



Disconnected Graph :-

A graph is said to be disconnected graph, if ∄ a path doesn't exist every pair of vertices in the graph



Working Rule :-

→ Both graphs have the same no. of vertices (v = v')

→ Both graphs have the same no. of edges [e = e']

→ Both graphs have onequal no. of vertices of 2 graphs

→ Both graphs have the graph there is corresponding vertix in 2 [every vertice in graphs]

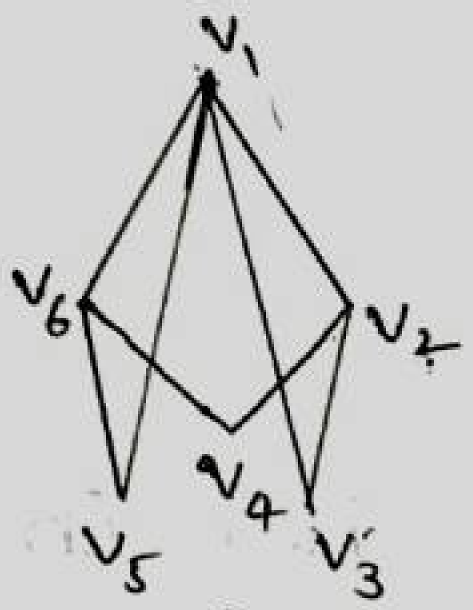
→ One to one correspondance the vertix of 2 graphs. [Every vertix in the first graph there is corresponding vertix in second graph]

→ Edge preserving is also satisfied b/w 2 graphs.

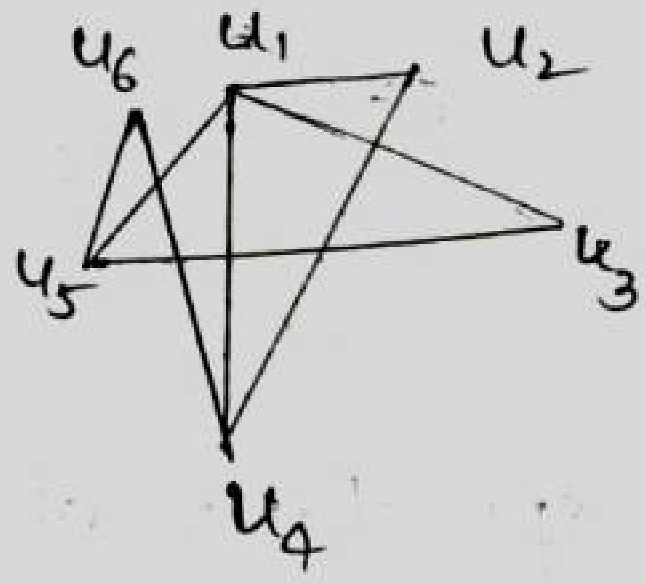
→ Smallest possible length is same for 2 graphs.

→ Adjacent matrix of 2 graphs should be same [G, G']

1) verify the following graph are isomorphic or not.



G(V, E)



G'(U, E').

No. of vertices (G) = 6

No. of vertices (G') = 6

No. of edges (G) = 8

No. of edges (G') = 8

Degree of vertices (G) = {4, 3, 2, 2, 2, 3}

Degree of vertices (G') = {4, 2, 2, 3, 3, 2}.

Vertices preserving:-

v_1 < v_2, v_3, v_5, v_6

v_2 < v_1, v_3, v_4

v_3 < v_1, v_2

v_4 < v_2, v_6

v_5 < v_1, v_6

v_6 < v_1, v_4, v_5

u_1 < u_2, u_3, u_4, u_5

u_2 < u_1, u_4

u_3 < u_1, u_5

u_4 < u_1, u_2, u_6

u_5 < u_1, u_3, u_6

u_6 < u_4, u_5

Edge preserving:-

$\{v_1, v_2\} = \{u_1, u_2\}$
$\{v_1, v_3\} = \{u_1, u_3\}$
$\{v_1, v_5\} = \{u_1, u_5\}$
$\{v_2, v_4\} = \{u_2, u_4\}$.
$\{v_4, v_6\} = \{u_4, u_6\}$
$\{v_5, v_6\} = \{u_5, u_6\}$.

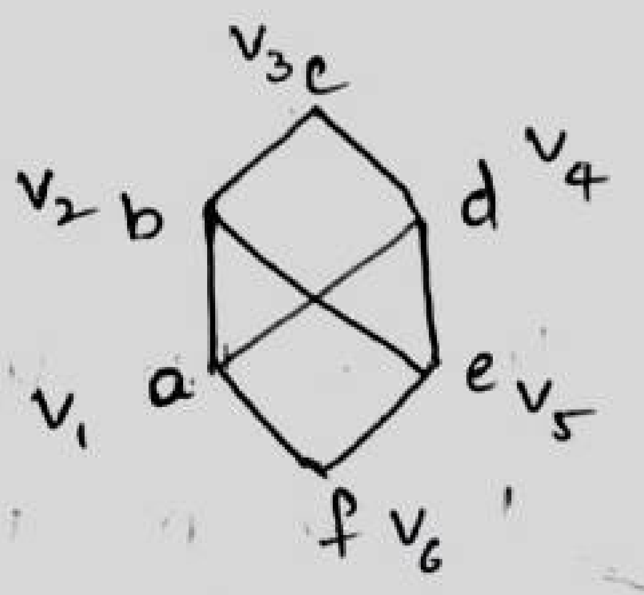
Adjacent matrix:-

$$\begin{array}{c|cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline v_1 & 0 & 1 & 1 & 0 & 1 & 1 \\ v_2 & 1 & 0 & 1 & 1 & 0 & 0 \\ v_3 & 1 & 1 & 0 & 0 & 0 & 0 \\ v_4 & 0 & 1 & 0 & 0 & 0 & 1 \\ v_5 & 1 & 0 & 0 & 0 & 0 & 1 \\ v_6 & 1 & 0 & 0 & 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 0 & 1 & 1 & 1 & 1 & 0 \\ u_2 & 1 & 0 & 0 & 1 & 0 & 0 \\ u_3 & 1 & 0 & 0 & 0 & 1 & 0 \\ u_4 & 1 & 1 & 0 & 0 & 0 & 1 \\ u_5 & 1 & 0 & 1 & 0 & 0 & 1 \\ u_6 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

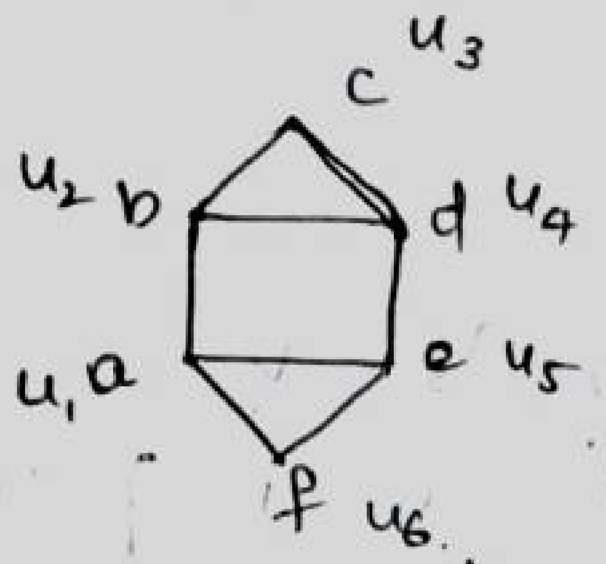
$\therefore \; G(V,E) \neq G'(U,E')$

$\therefore$ The given graphs aren't isomorphic.

2) Determine whether the given graphs are isomorphic



G(V, E)



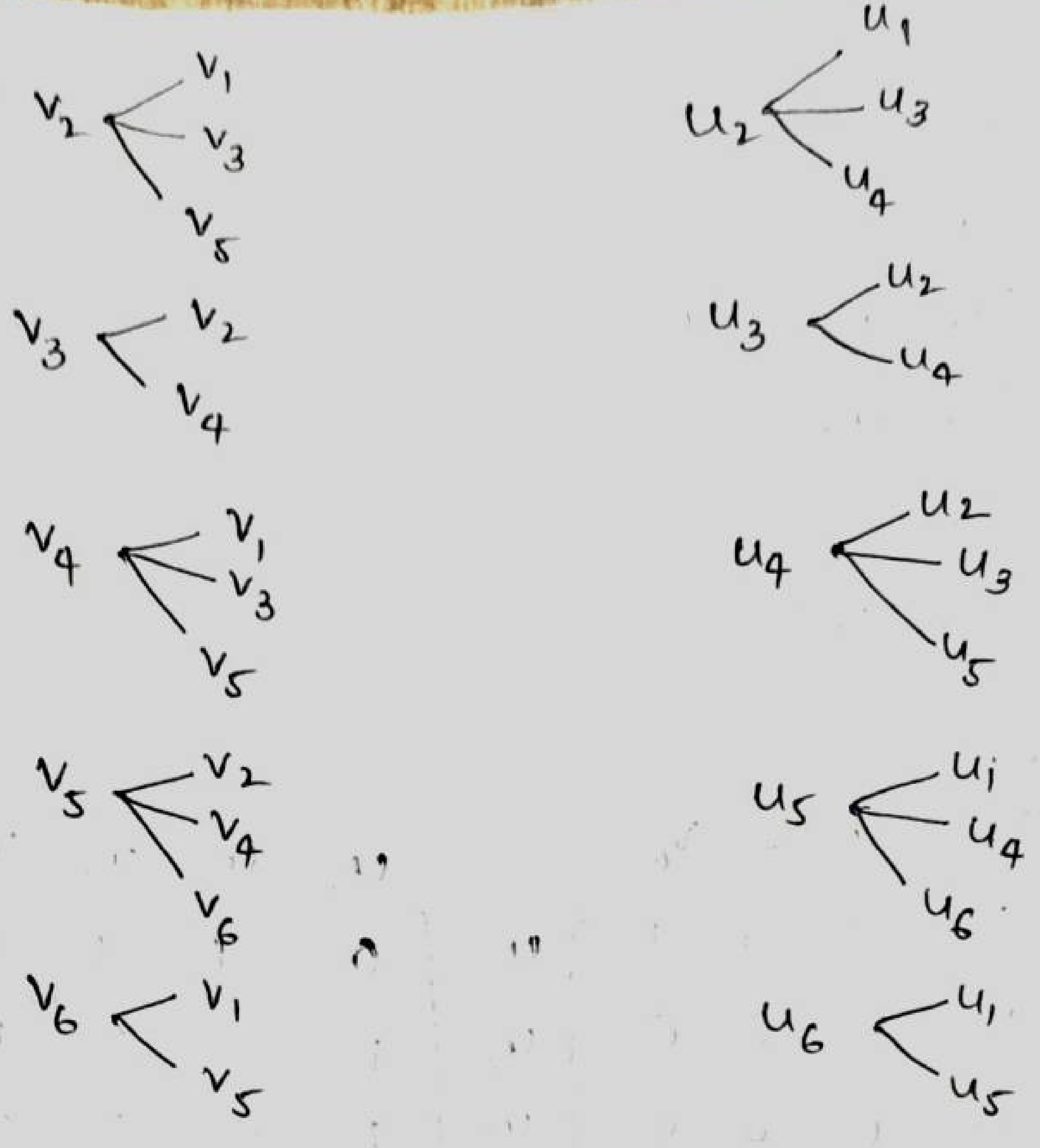
H (U, E').

No. of vertices (G) = 6.
No. of vertices (H) = 6
No. of edges (G) = 8
No. of edges (H) = 8.

degree of vertices (G) = {3, 3, 2, 3, 3, 2}
degree of vertices (H) = {3, 3, 2, 3, 3, 2}

Vertices preserving:-

$v_1 < v_2, v_4, v_6$ $\qquad$ $u_1 < u_2, u_5, u_6$



Edge preserving :-

$\{v_1, v_2\} = \{u_1, u_2\}$.

$\{v_1, v_6\} = \{u_1, u_6\}$

$\{v_2, v_3\} = \{u_2, u_3\}$

$\{v_3, v_4\} = \{u_3, u_4\}$

$\{v_4, v_5\} = \{u_4, u_5\}$

$\{v_5, v_6\} = \{u_5, u_6\}$

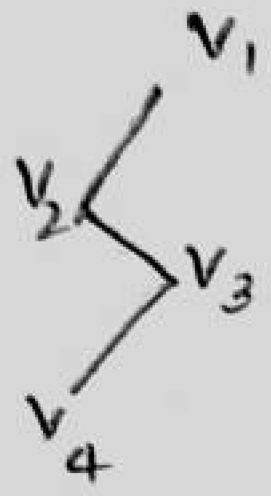
Adjacent matrix :-

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \quad \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

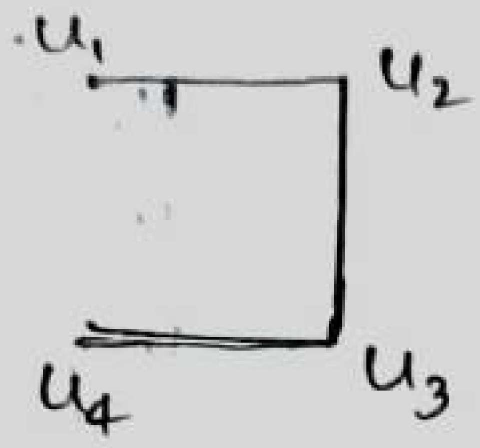
$\therefore G(V, E) \neq H(U, E')$

$\therefore$ The given graph isn't isomorphic.

3) Verify the following graphs are isomorphic or not.



$G(V,E)$



$H(U,E')$.

NO. of vertices $(G) = 4$

NO. of vertices $(H) = 4$.

NO. of edges $(G) = 3$

NO. of edges $(H) = 3$.

degree of vertices $(G) = \{1,2,2,1\}$.

degree of vertices $(H) = \{1,2,2,1\}$.

Vertices preserving:-

$v_1 - v_2$.

$v_2 < \begin{matrix} v_1 \\ v_3 \end{matrix}$

$v_3 < \begin{matrix} v_2 \\ v_4 \end{matrix}$

$v_4 - v_3$

$u_1 - u_2$

$u_2 < \begin{matrix} u_1 \\ u_3 \end{matrix}$

$u_3 < \begin{matrix} u_2 \\ u_4 \end{matrix}$

$u_4 - u_3$.

Edge preserving:-

$\{v_1, v_2\} = \{u_1, u_2\}$

$\{v_2, v_3\} = \{u_2, u_3\}$

$\{v_3, v_4\} = \{u_3, u_4\}$.

Adjacent vertices:-

$$\begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 0 & 0 \\ v_2 & 1 & 0 & 1 & 0 \\ v_3 & 0 & 1 & 0 & 1 \\ v_4 & 0 & 0 & 1 & 0 \end{array} \quad ; \quad \begin{array}{c|cccc} & u_1 & u_2 & u_3 & u_4 \\ \hline u_1 & 0 & 1 & 0 & 0 \\ u_2 & 1 & 0 & 1 & 0 \\ u_3 & 0 & 1 & 0 & 1 \\ u_4 & 0 & 0 & 1 & 0 \end{array}$$

$\therefore G(V,E) = H(U,E')$

$\therefore$ The given graph is isomorphic.

Eular circlet:-

A Closed walk (or) path which is visited every edge of the graph exactly ones.

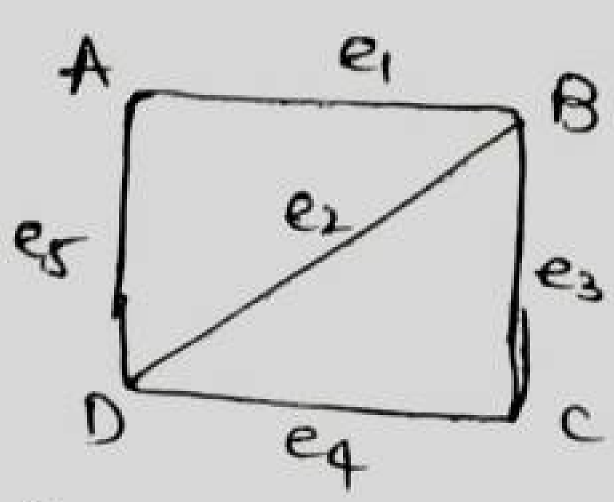
Eular (or) Eularian graph:-

A graph G is said to be Eular graph if and only if it Contains a Eular circuit.

$\boxed{\text{Eular}}$
↓
$\boxed{\text{Connected graph}}$
↓
$\boxed{\text{visited all the edge of the graph exactly once.}}$
↓
$\boxed{\text{It is a closed path}}$

1) Find the following graph is Eular graph or not.

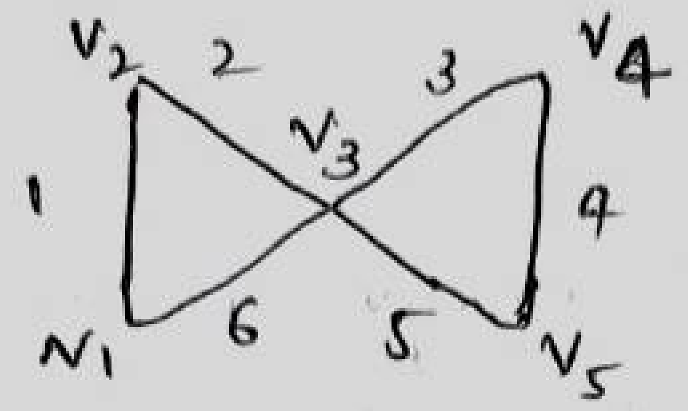
i)



$A \xrightarrow{e_1} B \xrightarrow{e_3} C \xrightarrow{e_4} D \xrightarrow{e_2} B.$

$\therefore$ It is not a closed path. and not a eular graph.

ii)



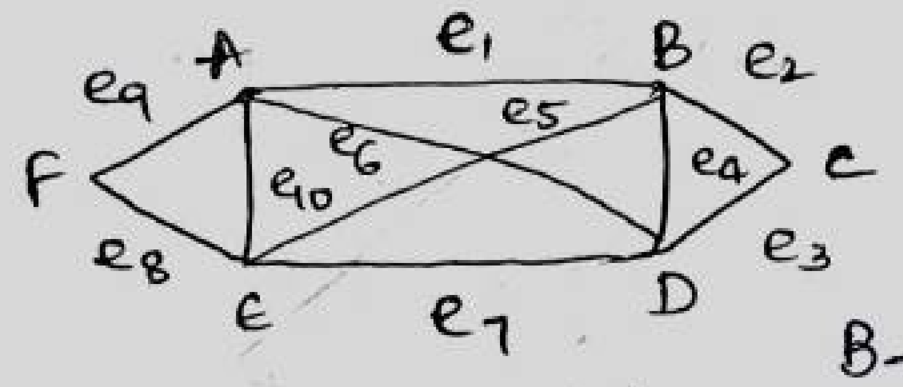
$V_1 \xrightarrow{1} V_2 \xrightarrow{2} V_3 \xrightarrow{3} V_4 \xrightarrow{4} V_5 \xrightarrow{5} V_3 \xrightarrow{6} V_1$.

∴ It is a closed path and visited by every edge exactly once

It is a connected graph.

∴ Given graph is Eular graph.

iii)



B e_5

$A \xrightarrow{e_1} B \xrightarrow{e_2} C \xrightarrow{e_3} D \xrightarrow{e_4} E \xrightarrow{e_8} F \xrightarrow{e_9} A \xrightarrow{e_{10}} E \xrightarrow{e_7} D \xrightarrow{e_6} A$

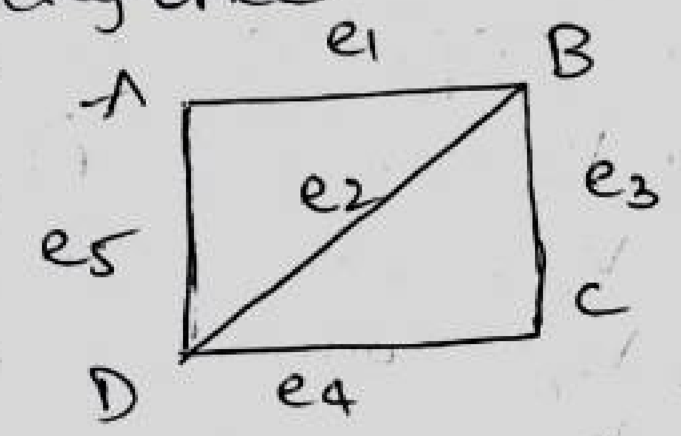
→ It is a closed path
→ It is visited by every edge exactly once.
→ It is a connected graph.

∴ Given graph is eular graph.

Semi-Eular graph:-

In a connected graph is an open path and it is visited by every edge exactly once

Ex:- i)

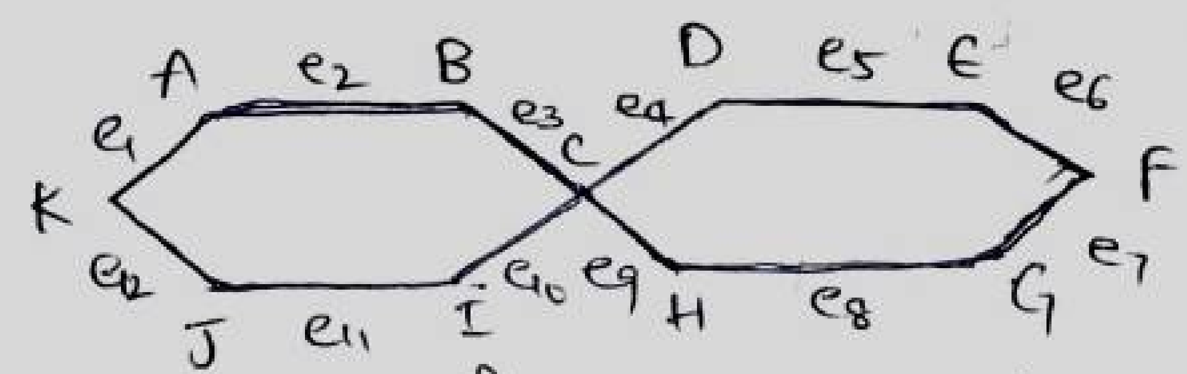


$B \xrightarrow{e_3} C \xrightarrow{e_4} D \xrightarrow{e_2} B \xrightarrow{e_1} A \xrightarrow{e_5} D$.

→ It is open path
→ It is visited by every edge exactly once.
→ It is a connected graph.

∴ Given graph is semi-eular graph.

ii)



~~e_3 B e_2 A e_1 K e_{12} J e_{11} I e_{10} C e_9 D~~
~~e_5 E e_6 F e_7 G e_8 H e_9 C~~

$A \xrightarrow{e_1} K \xrightarrow{e_{12}} J \xrightarrow{e_{11}} I \xrightarrow{e_{10}} C \xrightarrow{e_9} D \xrightarrow{e_5} E \xrightarrow{e_6} F \xrightarrow{e_7} G \xrightarrow{e_{10}} H \xrightarrow{e_9} C \xrightarrow{e_3} B \xrightarrow{e_2} A$

Find the following graph is eular or Semi-eular graph

→ It is a closed path
→ It visited by every edge exactly once.
→ It is a connected graph ∴ It is a eular graph.

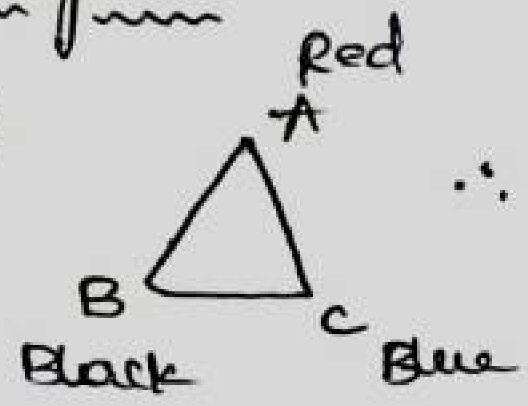
Graph Coloring:-

→ Given a planar (or) non-planar graph if we assign colors to its vertices in such a way that know to adjacent vertices have the same color. then we can say that the graph G is properly colored.

→ proper coloring of a graph means assigning colors of its vertices such that adjacent vertices are colored.

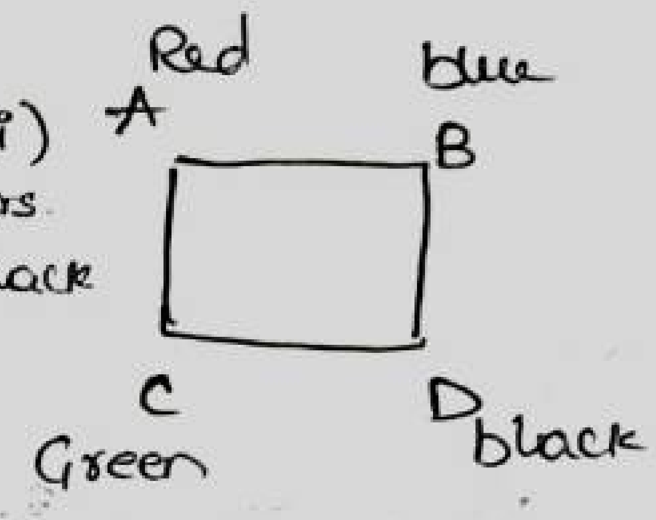
Eg:- 1

i)

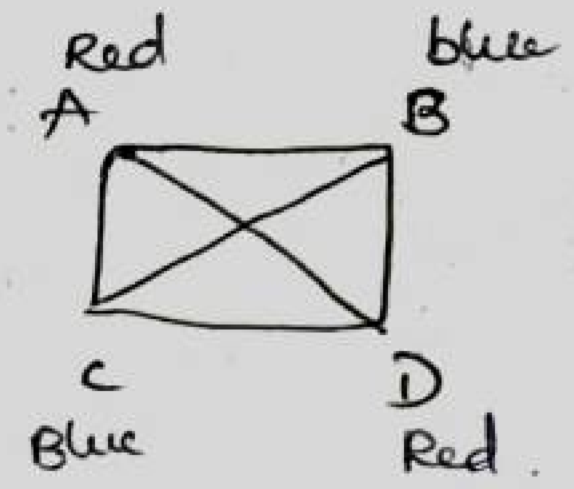


∴ using 3 colors red, blue, Black

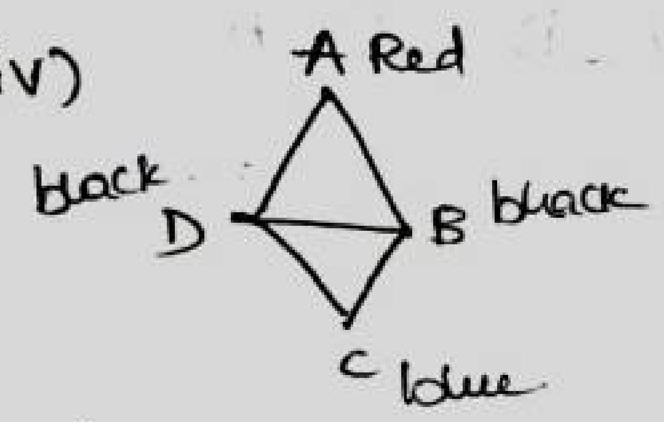
ii)



iii)

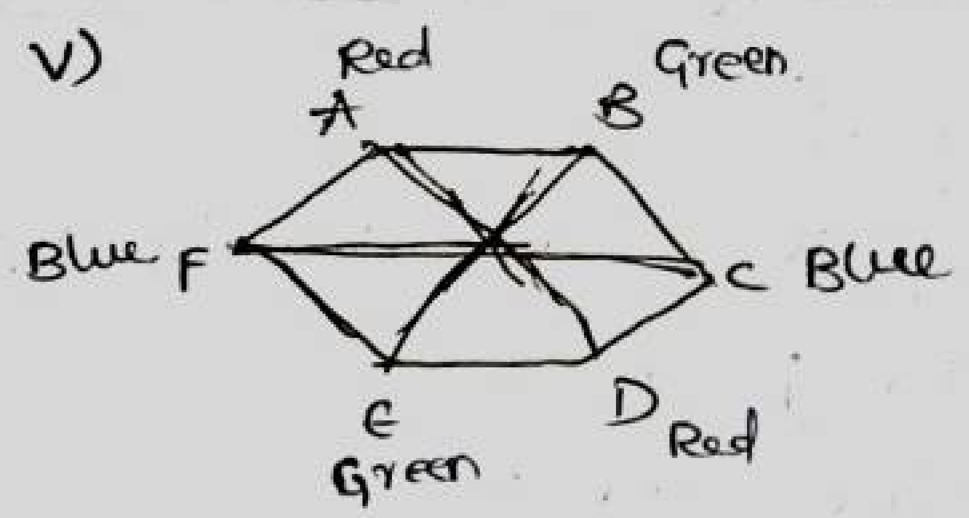


iv)



∴ using 3 colors Red, black, blue

v)



∴ using 3 colors red, Green, blue.

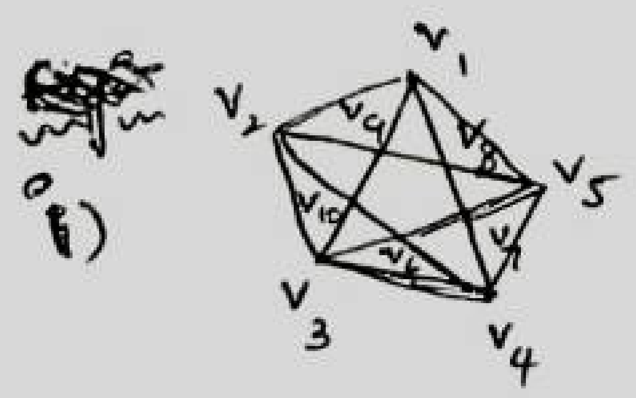
Chromatic number:-

→ The no. of colors req. to all the vertices of a given graph is called the chromatic no. of a given graph.

→ The chromatic number of a graph 'G' is denoted by '$x(G)$'

1) find the chromatic number of the following graph?.

i)



v_1 - Red v_4 - Green v_7 - red

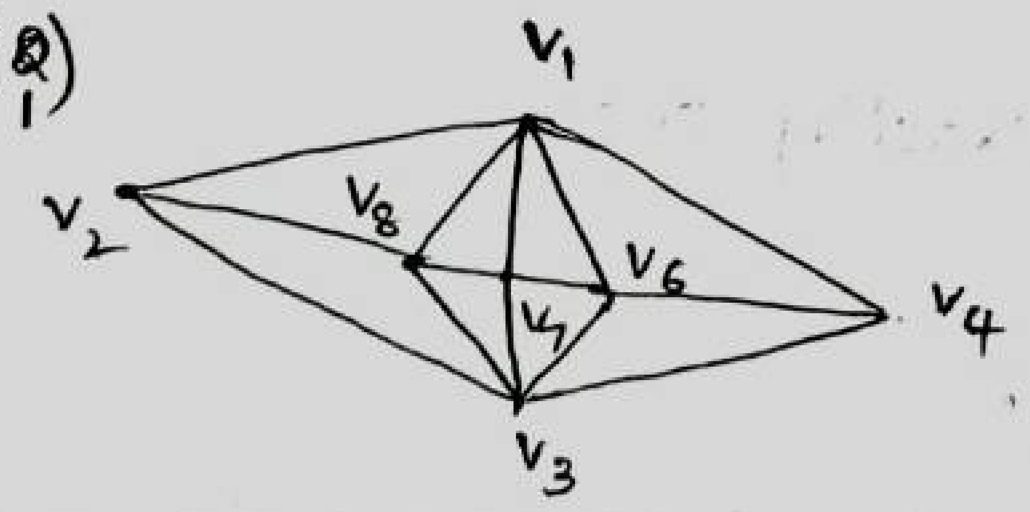
v_2 - Green v_5 - Blue v_8 - Green

v_3 - Red v_6 - Blue v_9 - Blue

v_{10} - purple.

∴ chromatic number $x(G) = 4$

ii)



v_1 - Red

v_2 - Blue v_6 - Green

v_3 - Red v_7 - Blue

v_4 - Blue v_8 - Green

∴ Chromatic number $x(G) = 4$.

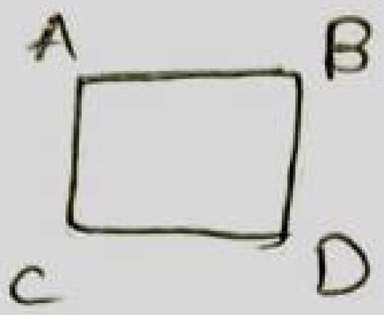
Ore's theorem :-

Proof :-

If G is a simple graph with n vertices ($n \geq 3$) such that for every pair of non-adjacent vertices u and v.

Degree of u + degree of $v \geq n$; where 'n' is the no. of vertices in a graph G.

Ex:- i)



i) $n \geq 3$

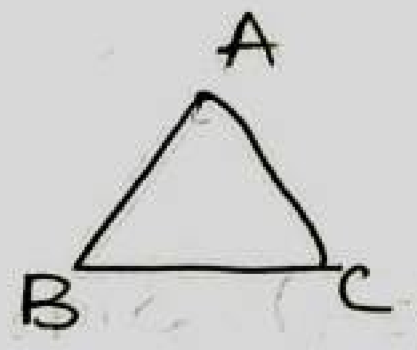
$\Rightarrow 4 \geq 3$ - true.

ii) Degree of A + Degree of $D \geq n$.

$2+2 \geq 4$

$4 \geq 4$.

ii)



i) $n \geq 3$

$3 \geq 3$ is true.

ii) Degree of 4 + Degree of $v \geq n$.

Hamilton path (or) walk :-

In a connected graph that visits every vertices exactly ones.

Hamilton circuit :-

First and last vertices are the same

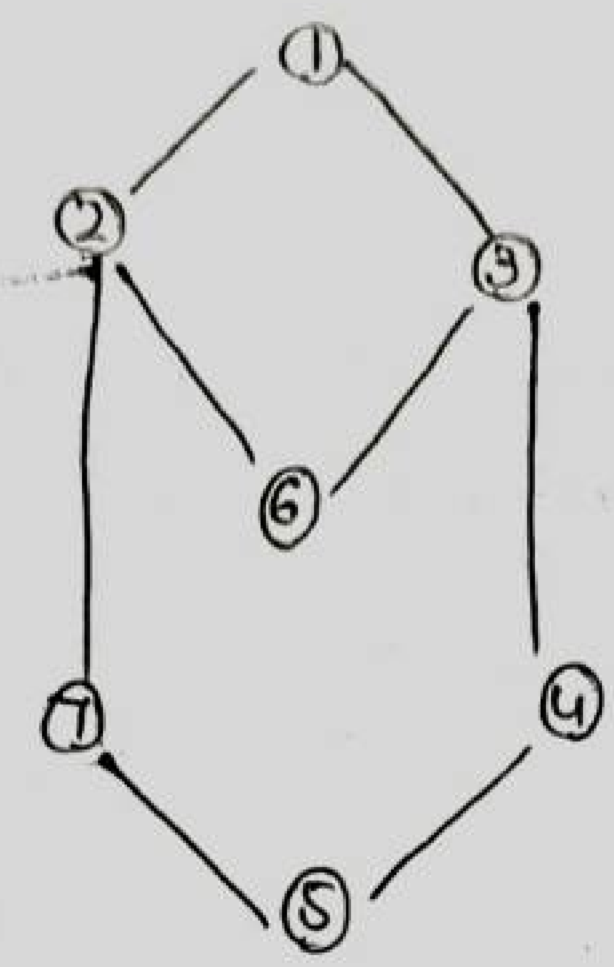
Hamilton graph :-

A graph G is said to be hamilton graph if it contains Hamilton circuit and Hamilton path.

Problems :-

1) Find the following graph is Hamilton or not :-

i)

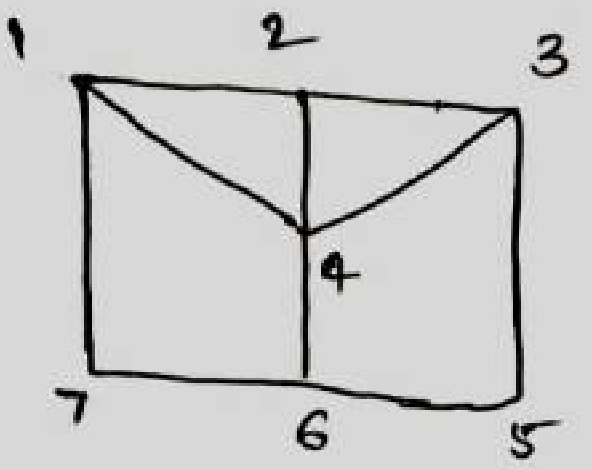


1 — 2 — 6 — 3 — 4 — 5

7 — 2

∴ It is not a Hamilton graph

ii)



1 — 2 — 4 — 3 — 5 — 6 — 7 — 1

→ It is a closed path

→ It is visited by every vertex exactly once.

→ It is a connected graph.

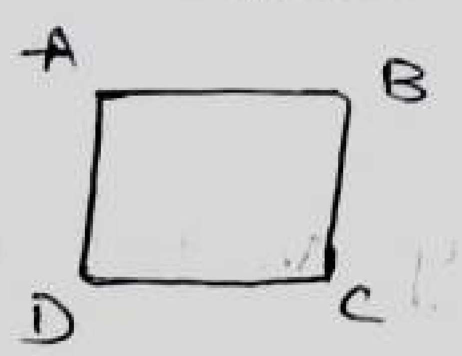
∴ It is a hamilton graph.

* Write about Dirac's theorem with proof :-

Statement:-

If G is a simple graph with 'n' vertices ($n \geq 3$) such that every vertex degree in 'G' is atleast $\frac{n}{2}$ then the graph 'G' has hamilton circuit where 'n' is a no. of vertices in a graph 'G'.

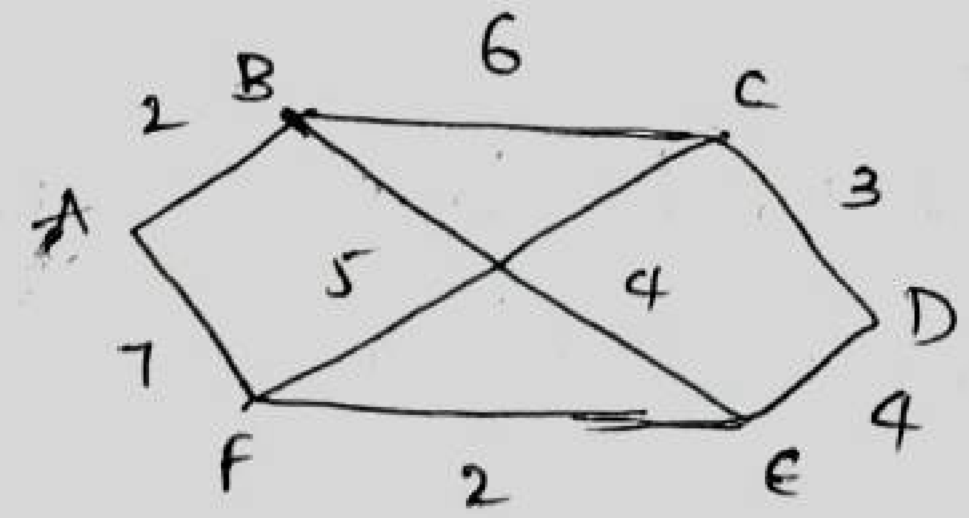
Eg:-

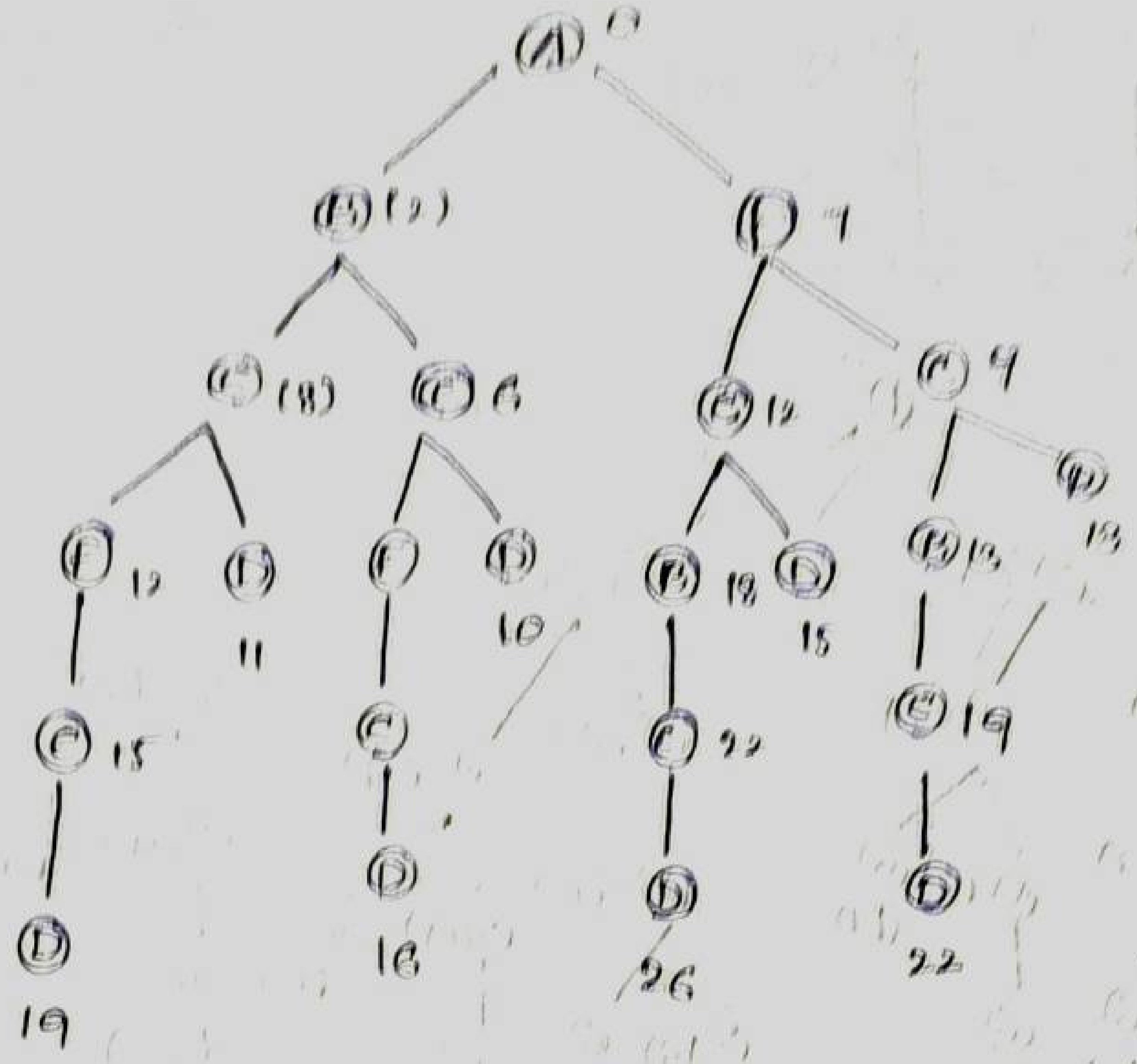


1. A — B — C — D — A
2. $n \geq 3 \Rightarrow 4 \geq 3$ true
3. $\frac{n}{2} \Rightarrow \frac{4}{2} = 2$.

1) Use Dijkstra's algorithm to find the length of the shortest path b/w the vertices A & D in the waited graphs shown below.

i)

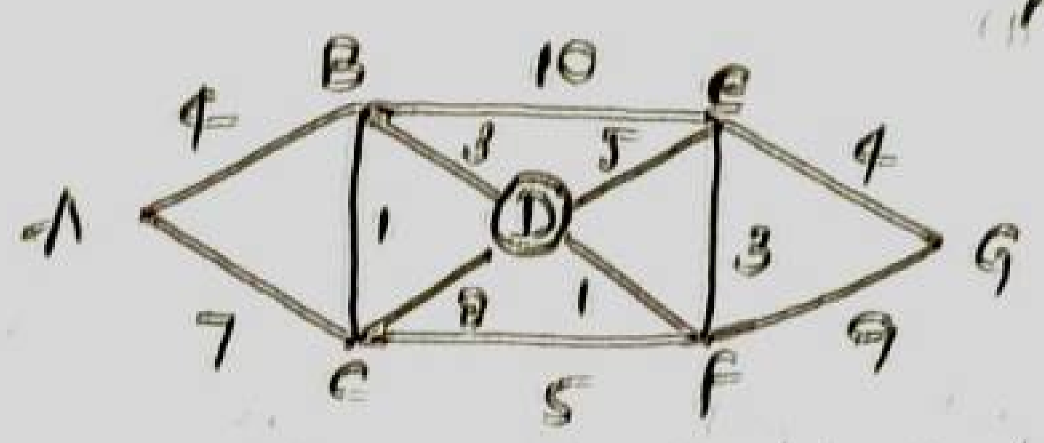


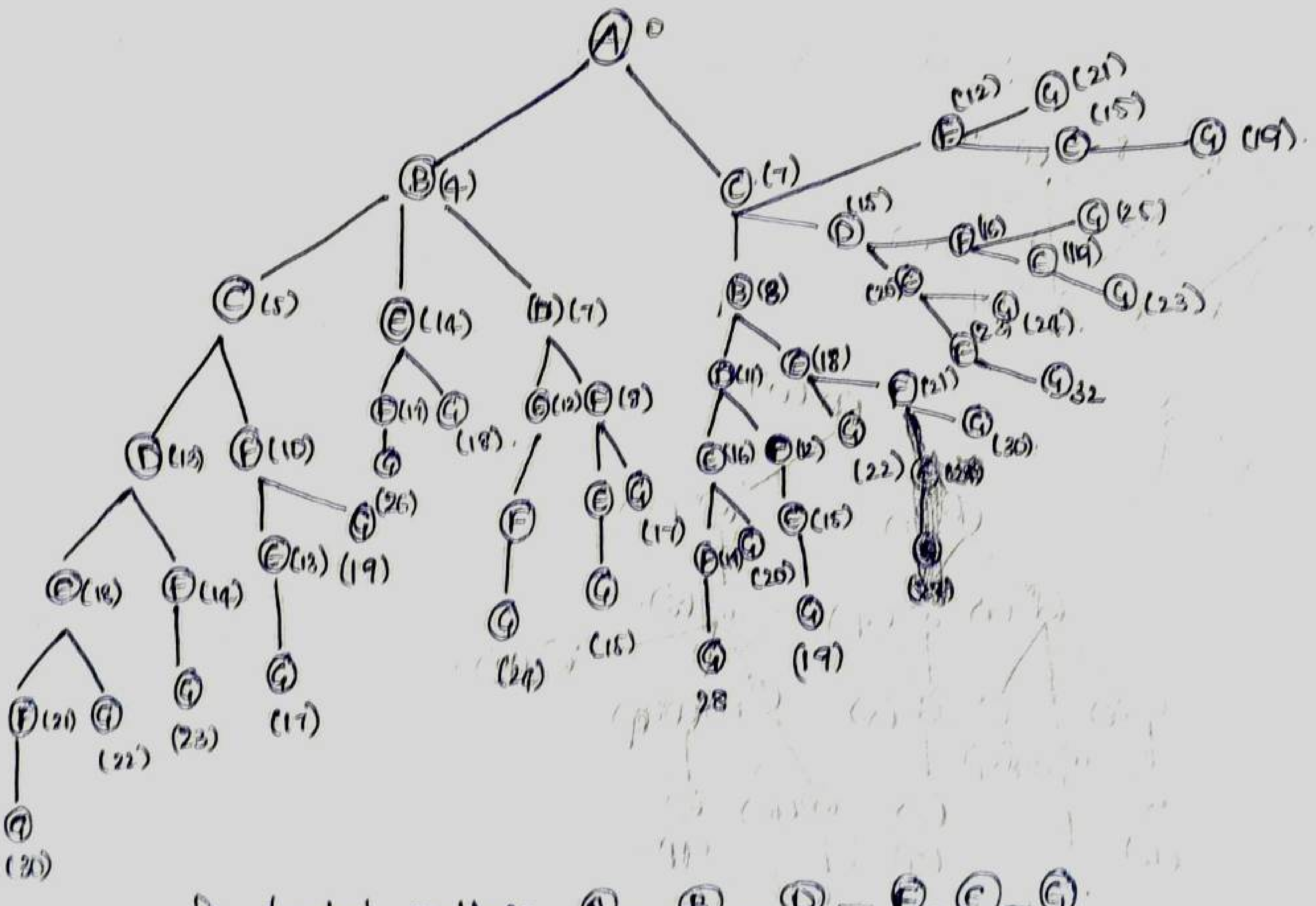


∴ the shortest path is A – B – C – D.

A & G (vertices).

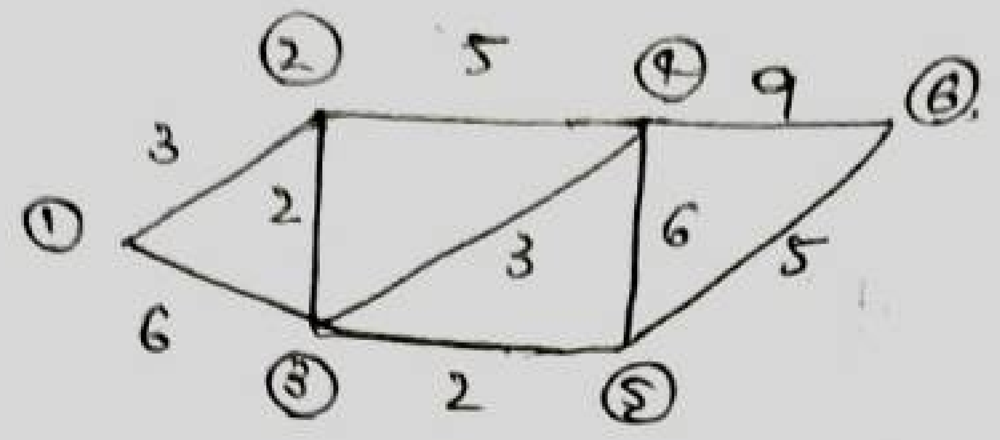
ii)

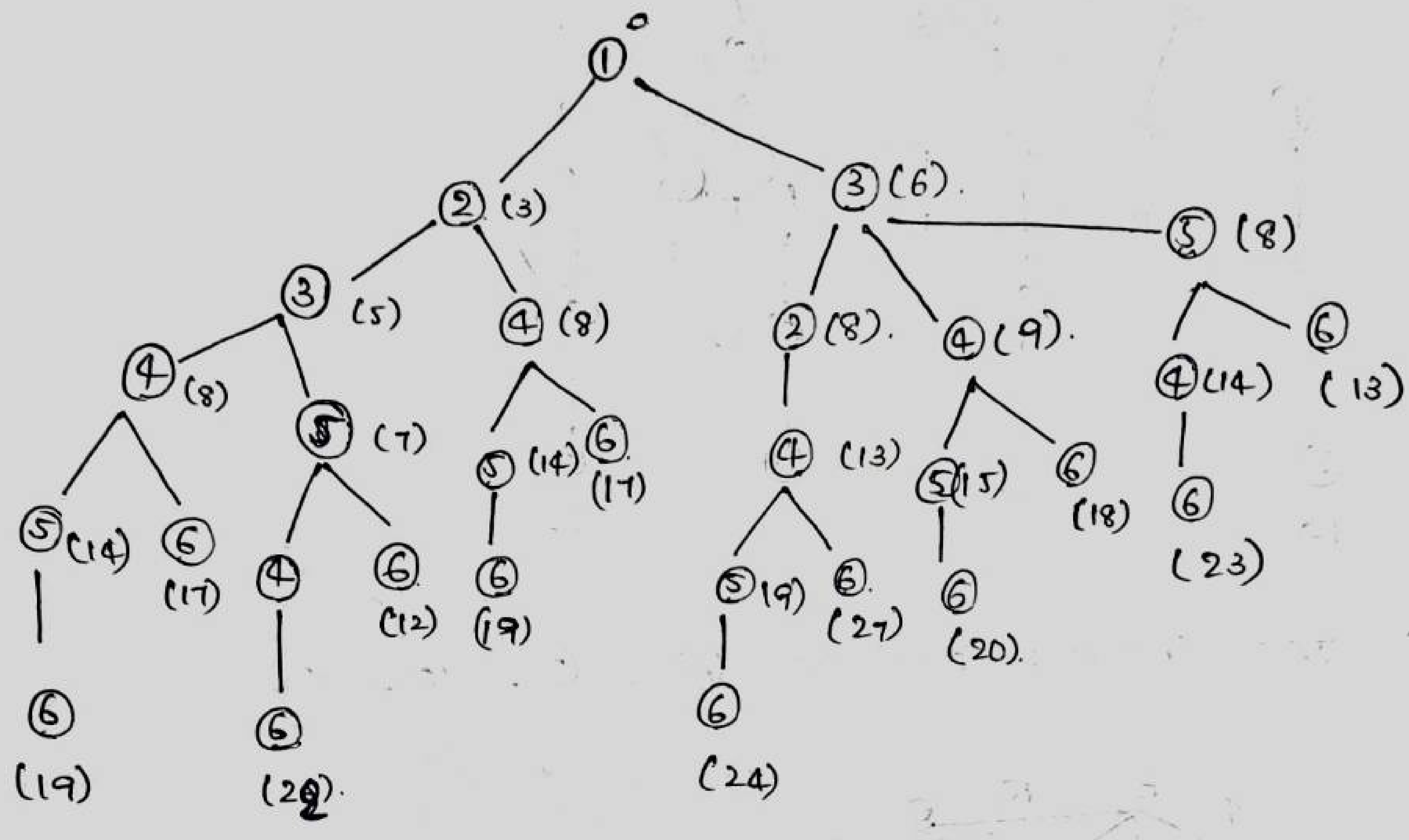




∴ the shortest path is A – B – D – F – C – G.

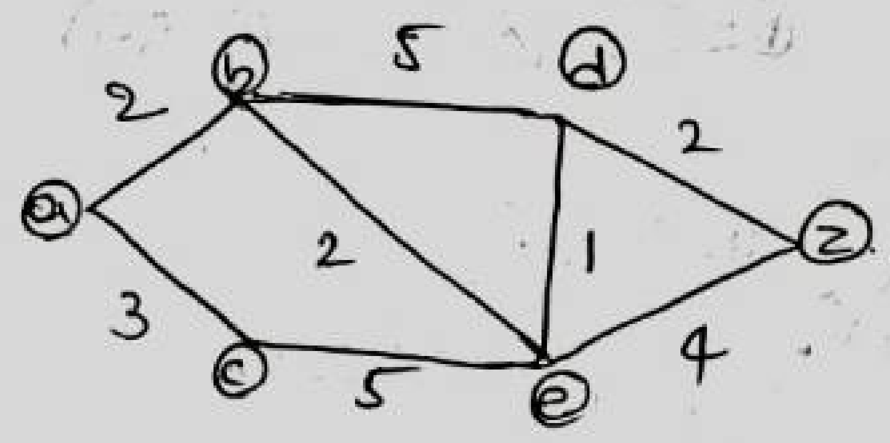
iii)

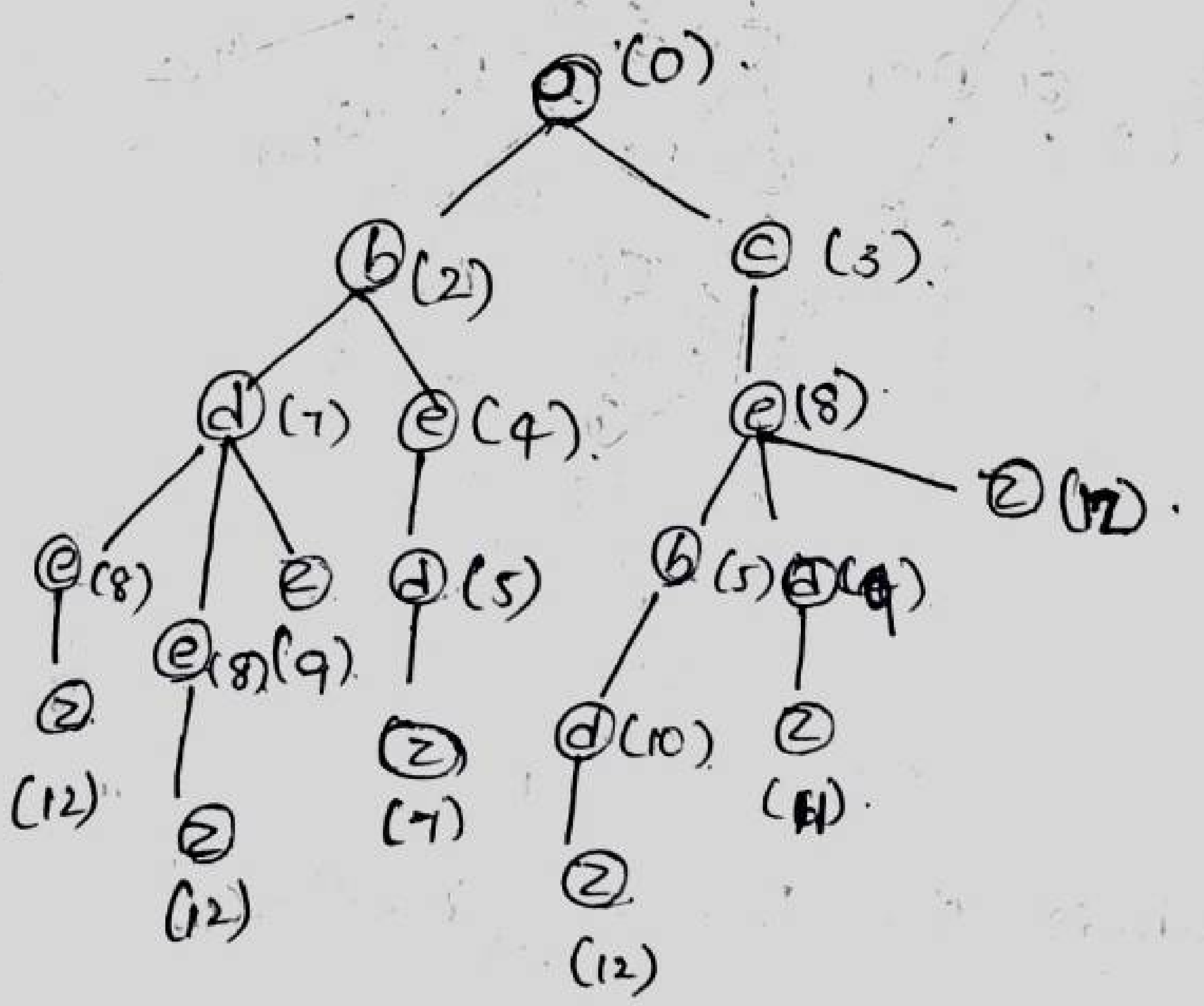




∴ The shortest path is 1 – 2 – 3 – 5 – 6

iv)





∴ The shortest path is a – b – e – d – z